This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

Educ T 128.93.575

3 2044 097 011 597

HARVARD COLLEGE
LIBRARY

GIFT OF THE
GRADUATE SCHOOL
OF EDUCATION

KEY

TO THE

HIGH SCHOOL ALGEBRA

CONTAINING SOLUTIONS OF THE MORE DIFFICULT EXAMPLES

BY

WILLIAM J. MILNE, PH.D., LL.D.

PRESIDENT OF NEW YORK STATE NORMAL COLLEGE, ALBANY, N.Y

NEW YORK .:. CINCINNATI .:. CHICAGO
AMERICAN BOOK COMPANY

Educ T 128.93.575

MILNE'S
MATHEMATICAL SERIES.

Elements of Arithmetic, . .	$0 30
Standard Arithmetic, . . .	65
First Lesson in Arithmetic, .	22
Elementary Arithmetic, . .	35
Practical Arithmetic, . . .	50
Inductive Algebra, ...	80
High School Algebra, . . .	1 00

Sent, postpaid, on receipt of price.

COPYRIGHT, 1893, BY
AMERICAN BOOK COMPANY.

ALL RIGHTS RESERVED.

KEY MIL. H. S. ALG.
E-P 14

HARVARD COLLEGE LIBRARY
GIFT OF THE
GRADUATE SCHOOL OF EDUCATION
OCT 11 1930

KEY

TO

HIGH SCHOOL ALGEBRA.

ALGEBRAIC EXPRESSIONS.

Page 18.

8. $(a+b)d-c=(1+2)\times 4-3=12-3=9.$

9. $(a+b)(d-c)=(1+2)\times(4-3)=3\times 1=3.$

10. $(a^2+b^2)\div(a+b)=(1+4)\div(1+2)=5\div 3=1\frac{2}{3}.$

11. $4(3a-b)=4(3-2)=4\times 1=4.$

12. $7a(3d-2a)=7(12-2)=7\times 10=70.$

13. $abcd(a+b+c+d)=1\times 2\times 3\times 4(1+2+3+4)=24(10)=240.$

14. $(a+b+c)(a+b+c)=(1+2+3)(1+2+3)=6\times 6=36.$

15. $(d+e-b)-(c-b)=(4+5-2)-(3-2)=7-1=6.$

16. $(d-a+c)(e-b)=(4-1+3)(5-2)=6\times 3=18.$

17. $(bc+d)-b(a+c)=(6+4)-2(1+3)=10-8=2.$

18. $(b^2-a+c)+(c-a)=(4-1+3)+(3-1)=6+2=8.$

19. $(c+d^2-a)\div(d-b)=(3+16-1)\div(4-2)=18\div 2=9.$

20. $2b(a+c)(e-c)=4(1+3)(5-3)=16\times 2=32.$

21. $3ad(c-a+d)-e^2=12(3-1+4)-25=72-25=47.$

22. $\frac{cd}{b}(a+b+c)-\sqrt{d}=\frac{12}{2}(1+2+3)-2=6\times 6-2=34.$

23. $\sqrt{5e}+a+\frac{3b}{2}=\sqrt{25}+1+\frac{6}{2}=5+1+3=9.$

24. $\left(\frac{a}{2}+\frac{c}{2}\right)d=\left(\frac{1}{2}+\frac{3}{2}\right)\times 4=2\times 4=8.$

25. $\frac{(a^2+b^2)3b}{2e+a}=\frac{(1+4)\times 6}{10+1}=\frac{5\times 6}{11}=\frac{30}{11}=2\frac{8}{11}.$

26. $a^2+b^2+c^2+d^2-e^2=1+4+9+16-25=5.$

27. $\sqrt{d}+(a+b)^2-e=\sqrt{4}+(1+2)^2-5=2+9-5=6.$

28. $(a+b)(b-a)\,4a = (1+2)\times(2-1)\times 4 = 3\times 1\times 4 = 12.$

29. $a + 3\sqrt{2e+\sqrt{4e+3d+2b}} = 1 + 3\sqrt{10+\sqrt{20+12+4}}$

$= 1 + 3\sqrt{10+6} = 1 + (3\times 4) = 13.$

30. $\left(\frac{ae}{a+c}+\frac{3d}{c}\right)^2 = \left(\frac{1\times 5}{4}+\frac{12}{3}\right)^2 = \left(\frac{21}{4}\right)^2 = \frac{441}{16} = 27\tfrac{9}{16}.$

31. $\frac{3c(a^3-c^3)-a}{2d+b} = \frac{9(64-27)-1}{8+2} = \frac{(9\times 37)-1}{10} = \frac{333-1}{10} = \frac{332}{10} = 33\tfrac{1}{5}.$

32. $\frac{b^2c^2}{a^2} - 5\sqrt{(a+c)^2} = 36 - 5\sqrt{16} = 36 - 20 = 16.$

33. $2b(a^2+c^2) - \sqrt{(d+e)} = 4(1+9) - \sqrt{9} = 40 - 3 = 37.$

34. $(a+d)^2 + ab\sqrt{(bd+a)} = 25 + 2\sqrt{8+1} = 25 + 6 = 31.$

35. $\left(\frac{4bc}{a+e}\right)^2 \div \frac{d^2}{b} = \left(\frac{24}{6}\right)^2 \div \frac{16}{2} = 16 \div 8 = 2.$

36. $\frac{a+c^2}{e} + \frac{de}{b+c} - \frac{(a+e)^2}{6} = \frac{1+9}{5} + \frac{20}{2+3} \quad \frac{(1+5)^2}{6} = 2 + 4 - 6 = 0.$

37. $4\,(abc - bc) + \frac{[\overline{a+b^2}+(c+d)^2]}{b+d} = 4(6-6) + \frac{5+49}{6} = 0 + \frac{54}{6} = 9.$

38. $\{[\overline{a+b}\times c\,(d+e) - a]\,b\} \div bd\sqrt{(a+d)^2}$

$= \{3\times 3\,(4+5) - 1\}2 \div 8\sqrt{(1+4)^2} = 160 \div 40 = 4.$

ADDITION—PROBLEMS.

Page 27.

14. Let x represent the amount he owes B;

then, $2x =$ the amount he owes C;

and, $10x =$ the amount due him.

Therefore, $10x - 3x = 1400$; (1)

$7x = 1400$; (2)

$x = 200$ dollars, the amount he owes B;

$2x = 400$ dollars, the amount he owes C.

15. Let x represent the number;

then, $13x - 5x + 8x = 155 + 5$; (1)

$16x = 160$; (2)

$x = 10$, the number.

16. Let x represent the number of books of fiction;

then, $10x =$ the number of books of reference;

and, $3x =$ the number of historical books.

Therefore, $10x - 3x - x = 12000$; (1)
$6x = 12000$; (2)
$x = 2000$, the books of fiction;
$10x = 20000$, the books of reference;
$3x = 6000$, the historical books.

17. Let x represent his indebtedness to B;
then, $10x =$ his indebtedness to A;
$3x =$ his indebtedness to C;
$4x =$ his indebtedness to D.
Therefore, $x + 10x + 3x + 4x = 36000$; (1)
$18x = 36000$; (2)
$x = 2000$ dollars, indebtedness to B;
$10x = 20000$ dollars, indebtedness to A;
$3x = 6000$ dollars, indebtedness to C.
$4x = 8000$ dollars, indebtedness to D.

18. Let x represent the number of votes for C;
then, $1\frac{1}{2}x =$ the number of votes for B;
and, $3x =$ the number of votes for A.
Therefore, $x + 1\frac{1}{2}x + 3x = 1125 - 3$; (1)
$5\frac{1}{2}x = 1122$; (2)
$x = 204$, the votes for C;
$1\frac{1}{2}x = 306$, the votes for B;
$3x = 612$, the votes for A.

19. Let x represent the sum paid for board per day;
then, $3x =$ the sum earned per day;
and, $15x =$ the sum earned in 5 days;
$9x =$ the sum paid for board in 9 days.
$15x - 9x = 24$; (1)
$6x = 24$; (2)
$x = 4$;
$9x = 36$ doll., the amount paid for board;
$15x = 60$ dollars, the amount earned.

20. Let $x =$ interest for 1 year;
then, $2x$, $3x$, $4x$, and $5x =$ different interests;

and $2x + 3x + 4x + 5x = 420$;

$14x = 420$;

$x = 30$; $2x = 60$ dollars, 1st; $3x = 90$ dollars, 2d;

$4x = 120$ dollars, 3d; $5x = 150$ dollars, 4th.

21. Let $x =$ the number of men;

$2x =$ the number of women;

$4x =$ the number of children;

then, $4x + 2x + x = 77$;

$7x = 77$;

$x = 11$, men; $2x = 22$, wom.; $4x = 44$, chil.

22. Let $x =$ sum given library;

$2x =$ sum given hospital;

$4x =$ sum given school;

then, $4x + 2x + x = 70000$;

$7x = 70000$;

$x = 10000$ dollars, sum given to library;

$2x = 20000$ dollars, sum given to hospital;

$4x = 40000$ dollars, sum given to school.

SIGNS OF AGGREGATION.

Page 34.

22. $2a - (2b - d) - \{a - b - (2c - 2d)\}$

$= 2a - 2b + d - \{a - b - 2c + 2d\}$

$= 2a - 2b + d - a + b + 2c - 2d$

$= a - b + 2c - d.$

23. $2a - [3b + (2b - c) - 4c + \{2a - (3b - \overline{c - 2b})\}]$

$= 2a - [3b + 2b - c - 4c + 2a - (3b - c + 2b)]$

$= 2a - 3b - 2b + c + 4c - 2a + 3b - c + 2b$

$= 4c.$

24. $2a - [2a - \{2a - (2a - \overline{2a - a})\}]$

$= 2a - 2a + \{2a - (2a - \overline{2a - a})\}$

$= 2a - 2a + \{2a - 2a + 2a - a\}$

$= 2a - 2a + 2a - 2a + 2a - a$

$= a.$

25. $x^2 - \{5\,mc^2 - [x - (3\,c - 3\,mc^2) + 3\,c - (x^2 - 2\,mc^2 - c)]\}$
$= x^2 - \{5\,mc^2 - [x - 3\,c + 3\,mc^2 + 3\,c - x^2 + 2\,mc^2 + c]\}$
$= x^2 - \{5\,mc^2 - x + 3\,c - 3\,mc^2 - 3\,c + x^2 - 2\,mc^2 - c\}$
$= x^2 - 5\,mc^2 + x - 3\,c + 3\,mc^2 + 3\,c - x^2 + 2\,mc^2 + c = x + c.$

26. $a - b - c - (d + 2\,a + [3\,b - 2\,c + d] - 4\,a - 2\,b)$
$= a - b - c - (d + 2\,a + 3\,b - 2\,c + d - 4\,a - 2\,b)$
$= a - b - c - d - 2\,a - 3\,b + 2\,c - d + 4\,a + 2\,b$
$= 3\,a - 2\,b + c - 2\,d.$

27. $x^2 + 2\,y - (x^3 + [2\,y + 3\,x^2 - 4\,x^3] - 6\,y + 3\,x^2) + 4\,x^3$
$= x^2 + 2\,y - (x^3 + 2\,y + 3\,x^2 - 4\,x^3 - 6\,y + 3\,x^2) + 4\,x^3$
$= x^2 + 2\,y - x^3 - 2\,y - 3\,x^2 + 4\,x^3 + 6\,y - 3\,x^2 + 4\,x^3$
$= -5\,x^2 + 7\,x^3 + 6\,y.$

28. $-(x^2y + 2\,y - 3) - (x^2y - [6\,y + 7 - 3\,x^2y] + 9)$
$= -x^2y - 2\,y + 3 - x^2y + [6\,y + 7 - 3\,x^2y] - 9$
$= -x^2y - 2\,y + 3 - x^2y + 6\,y + 7 - 3\,x^2y - 9$
$= -5\,x^2y + 4\,y + 1.$

29. $ab + bc - (3\,ab + [3\,bc + 2\,bd - 3\,ab] + 2\,bd) - 6\,c$
$= ab + bc - (3\,ab + 3\,bc + 2\,bd - 3\,ab + 2\,bd) - 6\,c$
$= ab + bc - 3\,ab - 3\,bc - 2\,bd + 3\,ab - 2\,bd - 6\,c$
$= ab - 2\,bc - 4\,bd - 6\,c.$

Page 35.

30. $-\{3\,ax - [2\,xy + 3\,z] + z - (4\,xy + [3\,ax + 6\,z] + 3\,z)\}$
$= -3\,ax + [2\,xy + 3\,z] - z + (4\,xy + [3\,ax + 6\,z] + 3\,z)$
$= -3\,ax + 2\,xy + 3\,z - z + 4\,xy + 3\,ax + 6\,z + 3\,z$
$= 6\,xy + 11\,z.$

31. $x - [-\{-(-x) + x\} - 2\,x] = x - [-\{x + x\} - 2\,x]$
$= x - [-x - x - 2\,x]$
$= x + x + x + 2\,x = 5\,x.$

32. $y^2 - (c^2 + x^2) - [y^2 - \{-(-\overline{x^2 + y^2})\}]$
$= y^2 - c^2 - x^2 - [y^2 - \{x^2 + y^2\}] = y^2 - c^2 - x^2 - [y^2 - x^2 - y^2]$
$= y^2 - c^2 - x^2 - y^2 + x^2 + y^2 = y^2 - c^2.$

33. $(a - b) - \{-a - (b - a) + (a - b)\} = a - b - \{-a - b + a + a - b\}$
$= a - b + a + b - a - a + b = b.$

34. $3\,a - (2\,a + 1) + \{a - (3 - \overline{4 - a})\} = 3\,a - 2\,a - 1 + \{a - 3 + 4 - a\}$
$= 3\,a - 2\,a - 1 + a - 3 + 4 - a = a.$

35. $-7-[-\{-a-(-a-\overline{a-3})\}]$
$=-7+\{-a-(-a-\overline{a-3})\}$
$=-7-a-(-a-\overline{a-3})$
$=-7-a+a+a-3=a-10.$

36. $(x^2+1)-[ax-\{-(-2ax+7)-\overline{ax-x^2-7}\}+2x^2]$
$=x^2+1-ax+\{-(-2ax+7)-\overline{ax-x^2-7}\}-2x^2$
$=x^2+1-ax+\{2ax-7-ax+x^2+7\}-2x^2$
$=x^2+1-ax+2ax-7-ax+x^2+7-2x^2=1.$

SUBTRACTION — PROBLEMS.

Page 39.

37. Let x represent what John had;
then, $x+10=$ what James had;
and, $x+15=$ what Henry had.
$\therefore\quad x+x+10+x+15=85.$ (1)
Transposing, $x+x+x=85-10-15;$ (2)
$3x=60;$ (3)
$x=20$ cents, the sum John had;
$x+10=30$ cents, the sum James had;
$x+15=35$ cents, the sum Henry had.

38. Let x represent the number in the first field;
then, $x+20=$ the number in the second field;
and, $x+45=$ the number in the third field.
$\therefore\quad x+x+20+x+45=395.$ (1)
Transposing, $x+x+x=395-20-45;$ (2)
$3x=330;$ (3)
$x=110$, the num. of sheep in the first field;
$x+20=130$, the num. of sheep in the second field;
$x+45=155$, the num. of sheep in the third field.

39. Let x represent the number he had;
therefore, $2x+4=100.$ (1)
Transposing, $2x=100-4;$ (2)
$2x=96;$ (3)
$x=48$, the number of cattle.

40. Let x represent the share of the youngest;

then, $x + 150 =$ the share of the next older;

$x + 300 =$ the share of the next older;

$x + 450 =$ the share of the oldest.

$\therefore \quad x+x+150+x+300+x+450 = 6900.$ (1)

Transposing, $x + x + x + x = 6900 - 150 - 300 - 450;$ (2)

$4x = 6900;$ (3)

$x = 1500$ dollars, the share of the youngest;

$x + 150 = 1650$ dollars, the share of the next older;

$x + 300 = 1800$ dollars, the share of the next older;

$x + 450 = 1950$ dollars, the share of the oldest.

41. Let x represent the expenses of the first year;

then, $x + 250 =$ the expenses of second year;

$x + 500 =$ the expenses of third year;

$x + 750 =$ the expenses of fourth year.

Therefore,

$x + x + 250 + x + 500 + x + 750 = 9500.$ (1)

Transposing, $x + x + x + x = 9500 - 250 - 500 - 750;$ (2)

$4x = 8000;$ (3)

$x = 2000$ dollars, the expenses of first year;

$x + 250 = 2250$ dollars, the expenses of second year;

$x + 500 = 2500$ dollars, the expenses of third year;

$x + 750 = 2750$ dollars, the expenses of fourth year.

MULTIPLICATION — EQUATIONS AND PROBLEMS.

Page 47.

10. $3x - 2(x + 1) = 13 - 7.$ (1)

Expanding, $3x - 2x - 2 = 13 - 7.$ (2)

Transposing, $3x - 2x = 13 - 7 + 2;$ (3)

$x = 8.$

11. $5x - 3(x - 4) = 4x + 7;$ (1)

Expanding, $5x - 3x + 12 = 4x + 7.$ (2)

Transposing, $5x - 3x - 4x = 7 - 12;$ (3)

$-2x = -5;$ (4)

$x = 2\frac{1}{2}.$

Page 48.

12. $4(x-5)-3(x+6)=0.$ (1)

Expanding, $4x-20-3x-18=0.$ (2)

Transposing, $4x-3x=20+18.$ (3)

$x=38.$ (4)

13. $(2+x)(x+3)=x^2+2x+18.$ (1)

Expanding, $x^2+5x+6=x^2+2x+18.$ (2)

Transposing, $x^2-x^2+5x-2x=18-6.$ (3)

$3x=12.$

$x=4.$

14. $5(2x-2)=27+3(2x+1).$ (1)

Expanding, $10x-10=27+6x+3.$ (2)

Transposing, $10x-6x=27+3+10.$ (3)

$4x=40.$ (4)

$x=10.$

15. $10(x-5)=(x+1)+5(x+1).$ (1)

Expanding, $10x-50=x+1+5x+5.$ (2)

Transposing, $10x-x-5x=1+5+50.$ (3)

$4x=56.$ (4)

$x=14.$

16. $5(x+3)-2(2x-7)=3(x-7).$ (1)

Expanding, $5x+15-4x+14=3x-21.$ (2)

Transposing, $5x-4x-3x=-15-14-21.$ (3)

$-2x=-50.$ (4)

$x=25.$

17. $3+7(x-2)-4(2x-7)=16+(x-2).$ (1)

Expanding, $3+7x-14-8x+28=16+x-2.$ (2)

Transposing, $7x-8x-x=16-2-28+14-3.$ (3)

$-2x=-3.$ (4)

$x=1\frac{1}{2}.$

18. $6x=15+3(x-3)-3(x-10).$ (1)

Expanding, $6x=15+3x-9-3x+30.$ (2)

Transposing, $6x-3x+3x=15+30-9.$ (3)

$6x=36.$ (4)

$x=6.$

19. $19 = 2(4 - x) + 5(7 + 2x) - 48.$ (1)

Expanding, $19 = 8 - 2x + 35 + 10x - 48.$ (2)

Transposing, $2x - 10x = 8 + 35 - 48 - 19.$ (3)

$-8x = -24.$ (4)

$x = 3.$

20. $2x + 3(6x - 5) - 5 = x - 1.$ (1)

Expanding, $2x + 18x - 15 - 5 = x - 1.$ (2)

Transposing, $2x + 18x - x = 15 + 5 - 1.$ (3)

$19x = 19.$ (4)

$x = 1.$

21. $3(x - 7) = 14 + 2(x - 10) + 2.$ (1)

Expanding, $3x - 21 = 14 + 2x - 20 + 2.$ (2)

Transposing, $3x - 2x = 14 - 20 + 2 + 21.$ (3)

$x = 17.$

23. Let x represent the number;

then, $x + 3(x + 2) = 22.$ (1)

Expanding, $x + 3x + 6 = 22.$ (2)

Transposing, $x + 3x = 22 - 6.$ (3)

$4x = 16.$ (4)

$x = 4.$

24. Let x represent B's age;

then, $2(x - 5) =$ A's age.

Therefore, $x + 2(x - 5) = 20.$ (1)

Expanding, $x + 2x - 10 = 20.$ (2)

Transposing, $x + 2x = 20 + 10.$ (3)

$3x = 30.$ (4)

$x = 10$, B's age.

$2(x - 5) = 10$, A's age.

25. Let x represent Henry's money;

then, $3(x - 5) =$ John's money.

Therefore, $x + 3(x - 5) = 21.$ (1)

Expanding, $x + 3x - 15 = 21.$ (2)

Transposing, $x + 3x = 21 + 15.$ (3)

$4x = 36.$ (4)

$x = 9$ cents, Henry's money.

$3(x - 5) = 12$ cents, John's money.

26. Let x represent the number of hours one traveled;

then, $x + 3 =$ the number of hours the other traveled;

and, $5x =$ the distance one traveled;

$5(x + 3) =$ the distance the other traveled.

Therefore, $5x + 5(x + 3) = 55.$ (1)

Expanding, $5x + 5x + 15 = 55.$ (2)

Transposing, etc., $10x = 40.$ (3)

$x = 4.$ (4)

$5x = 20$, the distance one traveled.

$5(x + 3) = 35$, the distance the other traveled.

Page 49.

27. Let x represent C's money;

then, $2x =$ B's money;

and, $4x =$ A's money.

Therefore, $(4x - 10) - (2x - 10) = x + 5;$ (1)

then, $4x - 10 - 2x + 10 = x + 5.$ (2)

Transposing, $4x - 2x - x = 10 + 5 - 10.$ (3)

$x = 5$ dollars, C's money.

$2x = 10$ dollars, B's money.

$4x = 20$ dollars, A's money.

28. Let x represent the acres in the smaller field;

then, $2(x + 10) =$ the acres in the larger field.

Therefore, $x + 2(x + 10) = 50.$ (1)

Expanding, $x + 2x + 20 = 50.$ (2)

Transposing, etc., $3x = 30.$ (3)

$x = 10$, the acres in the smaller field.

$2(x + 10) = 40$, the acres in the larger field.

29. Let x represent B's capital;

then, $2x =$ A's capital.

Therefore, $2x + 500 = 3(x - 300).$ (1)

Expanding, $2x + 500 = 3x - 900.$ (2)

Transposing, etc., $x = 1400$ dollars, B's capital.

$2x = 2800$ dollars, A's capital.

30. Let x represent the quantity he wanted;

then, $x - 6 =$ the quantity in the first;

and, $x + 14 =$ the quantity in the other.

Therefore, $3(x - 6) = 2(x + 14)$. (1)

Expanding, $3x - 18 = 2x + 28$. (2)

Transposing, $x = 46$, the quantity he wanted;

$x - 6 = 40$ pounds, the quantity in the first;

$x + 14 = 60$ pounds, the quantity in the other.

31. Let $x =$ cost;

$\frac{x}{10} =$ share of each;

then, $\frac{x}{6} - \frac{x}{10} = 8$; (1)

$5x - 3x = 240$; (2)

$2x = 240$;

$x = 120$ dollars, cost.

32. Let $x =$ distance of 1st;

$49 - x =$ distance of 2d;

then, $\frac{x}{4} + 2 = \frac{49 - x}{5} + 3$; (1)

$5x + 40 = 196 - 4x + 60$; (2)

$9x = 216$;

$x = 24$ mi. 1st traveled;

and $49 - x = 25$ mi. 2d traveled.

33. Let $x =$ number of doz. at 25¢;

$13 - x =$ number of doz. at 30¢;

then, $25x + 390 - 30x = 350$; (1)

$5x = 40$; $x = 8$; and $13 - x = 5$. (2)

34. Let $x =$ B's age;

then, $3x =$ A's age;

and, $3x - 8 =$ A's age 8 yr. ago;

$x - 8 =$ B's age 8 yr. ago.

Therefore, $3x - 8 = 7(x - 8)$. (1)

Expanding, $3x - 8 = 7x - 56$; (2)

whence, $x = 12$, B's age;

and $3x = 36$, A's age.

SPECIAL CASES IN MULTIPLICATION.

Page 51.

24. $22^2 = (20 + 2)^2 = 20^2 + 2 \times 20 \times 2 + 2^2 = 484.$
25. $23^2 = (20 + 3)^2 = 20^2 + 2 \times 20 \times 3 + 3^2 = 529.$
26. $41^2 = (40 + 1)^2 = 40^2 + 2 \times 40 \times 1 + 1^2 = 1681.$
27. $52^2 = (50 + 2)^2 = 50^2 + 2 \times 50 \times 2 + 2^2 = 2704.$
28. $71^2 = (70 + 1)^2 = 70^2 + 2 \times 70 \times 1 + 1^2 = 5041.$
29. $82^2 = (80 + 2)^2 = 80^2 + 2 \times 80 \times 2 + 2^2 = 6724.$
30. $91^2 = (90 + 1)^2 = 90^2 + 2 \times 90 \times 1 + 1^2 = 8281.$
31. $101^2 = (100 + 1)^2 = 100^2 + 2 \times 100 \times 1 + 1^2 = 10201.$
32. $103^2 = (100 + 3)^2 = 100^2 + 2 \times 100 \times 3 + 3^2 = 10609.$
33. $202^2 = (200 + 2)^2 = 200^2 + 2 \times 200 \times 2 + 2^2 = 40804.$
34. $207^2 = (200 + 7)^2 = 200^2 + 2 \times 200 \times 7 + 7^2 = 42849.$
35. $303^2 = (300 + 3)^2 = 300^2 + 2 \times 300 \times 3 + 3^2 = 91809.$
37. $(5\frac{1}{2})^2 = 5 \times 6 + \frac{1}{4} = 30\frac{1}{4}.$
38. $(8\frac{1}{2})^2 = 8 \times 9 + \frac{1}{4} = 72\frac{1}{4}.$
39. $(7\frac{1}{2})^2 = 7 \times 8 + \frac{1}{4} = 56\frac{1}{4}.$
40. $(12\frac{1}{2})^2 = 12 \times 13 + \frac{1}{4} = 156\frac{1}{4}.$
41. $(10\frac{1}{2})^2 = 10 \times 11 + \frac{1}{4} = 110\frac{1}{4}.$
42. $(9\frac{1}{2})^2 = 9 \times 10 + \frac{1}{4} = 90\frac{1}{4}.$
43. $2.5^2 = 2 \times 3 + .25 = 6.25.$
44. $3.5^2 = 3 \times 4 + .25 = 12.25$
45. $4.5^2 = 4 \times 5 + .25 = 20.25.$
46. $45^2 = (4.5 \text{ tens})^2 = 4 \times 5 \text{ tens} + 25 = 2025.$
47. $7.5^2 = 7 \times 8 + .25 = 56.25.$
48. $75^2 = (7.5 \text{ tens})^2 = 7 \times 8 \text{ tens} + 25 = 5625.$
49. $8.5^2 = 8 \times 9 + .25 = 72.25.$
50. $85^2 = (8.5 \text{ tens})^2 = 8 \times 9 \text{ tens} + 25 = 7225.$
51. $35^2 = (3.5 \text{ tens})^2 = 3 \times 4 \text{ tens} + 25 = 1225.$

Page 52.

24. $18^2 = (20 - 2)^2 = 20^2 - 2 \times 20 \times 2 + 2^2 = 324.$
25. $29^2 = (30 - 1)^2 = 30^2 - 2 \times 30 \times 1 + 1^2 = 841.$
26. $39^2 = (40 - 1)^2 = 40^2 - 2 \times 40 \times 1 + 1^2 = 1521.$
27. $38^2 = (40 - 2)^2 = 40^2 - 2 \times 40 \times 2 + 2^2 = 1444.$
28. $49^2 = (50 - 1)^2 = 50^2 - 2 \times 50 \times 1 + 1^2 = 2401.$
29. $48^2 = (50 - 2)^2 = 50^2 - 2 \times 50 \times 2 + 2^2 = 2304.$
30. $59^2 = (60 - 1)^2 = 60^2 - 2 \times 60 \times 1 + 1^2 = 3481.$
31. $58^2 = (60 - 2)^2 = 60^2 - 2 \times 60 \times 2 + 2^2 = 3364.$
32. $79^2 = (80 - 1)^2 = 80^2 - 2 \times 80 \times 1 + 1^2 = 6241.$

33. $78^2 = (80-2)^2 = 80^2 - 2 \times 80 \times 2 + 2^2 = 6084.$

34. $99^2 = (100-1)^2 = 100^2 - 2 \times 100 \times 1 + 1^2 = 9801.$

35. $98^2 = (100-2)^2 = 100^2 - 2 \times 100 \times 2 + 2^2 = 9604.$

36. $997^2 = (1000-3)^2 = 1000^2 - 2 \times 1000 \times 3 + 3^2 = 994009.$

37. $998^2 = (1000-2)^2 = 1000^2 - 2 \times 1000 \times 2 + 2^2 = 996004.$

38. $999^2 = (1000-1)^2 = 1000^2 - 2 \times 1000 \times 1 + 1^2 = 998001.$

Page 54.

35. $19 \times 21 = (20-1)(20+1) = 400 - 1 = 399.$

36. $29 \times 31 = (30-1)(30+1) = 900 - 1 = 899.$

37. $33 \times 27 = (30+3)(30-3) = 900 - 9 = 891.$

38. $56 \times 64 = (60-4)(60+4) = 3600 - 16 = 3584.$

39. $74 \times 66 = (70+4)(70-4) = 4900 - 16 = 4884.$

40. $89 \times 91 = (90-1)(90+1) = 8100 - 1 = 8099.$

41. $78 \times 82 = (80-2)(80+2) = 6400 - 4 = 6396.$

42. $97 \times 103 = (100-3)(100+3) = 10000 - 9 = 9991.$

43. $34 \times 26 = (30+4)(30-4) = 900 - 16 = 884.$

44. $38 \times 42 = (40-2)(40+2) = 1600 - 4 = 1596.$

45. $57 \times 63 = (60-3)(60+3) = 3600 - 9 = 3591.$

46. $45 \times 55 = (50-5)(50+5) = 2500 - 25 = 2475.$

47. $99 \times 101 = (100-1)(100+1) = 10000 - 1 = 9999.$

48. $98 \times 102 = (100-2)(100+2) = 10000 - 4 = 9996.$

49. $96 \times 104 = (100-4)(100+4) = 10000 - 16 = 9984.$

50. $94 \times 106 = (100-6)(100+6) = 10000 - 36 = 9964.$

53. $19^2 = (19+1)(19-1) + 1^2 = 361.$

54. $29^2 = (29+1)(29-1) + 1^2 = 841.$

55. $31^2 = (31+1)(31-1) + 1^2 = 961.$

56. $39^2 = (39+1)(39-1) + 1^2 = 1521.$

57. $59^2 = (59+1)(59-1) + 1^2 = 3481.$

58. $49^2 = (49+1)(49-1) + 1^2 = 2401.$

59. $98^2 = (98+2)(98-2) + 2^2 = 9604.$

60. $96^2 = (96+4)(96-4) + 4^2 = 9216.$

61. $94^2 = (94+6)(94-6) + 6^2 = 8836.$

62. $78^2 = (78+2)(78-2) + 2^2 = 6084.$

63. $79^2 = (79+1)(79-1) + 1^2 = 6241.$

64. $68^2 = (68+2)(68-2) + 2^2 = 4624.$

65. $103^2 = (103-3)(103+3) + 3^2 = 10609.$

66. $107^2 = (107-7)(107+7)+7^2 = 11449.$

67. $112^2 = (112-12)(112+12)+12^2 = 12544.$

68. $997^2 = (997+3)(997-3)+3^2 = 994009.$

DIVISION.

Page 63.

18.
$$\begin{array}{l|l}
a^2+2bc-b^2-c^2 & a-b+c \\ \hline
a^2-ab+ac & a+b-c \\ \hline
\quad ab-ac+2bc-b^2-c^2 & \\
\quad \underline{ab\qquad+bc-b^2} & \\
\qquad -ac+bc-c^2 & \\
\qquad \underline{-ac+bc-c^2} &
\end{array}$$

20.
$$\begin{array}{l|l}
ax^3-a^2x^2-bx^2+b^2 & ax-b \\ \hline
ax^3\qquad\quad -bx^2 & x^2-ax-b \\ \hline
-a^2x^2+b^2 & \\
\underline{-a^2x^2+abx} & \\
-abx+b^2 & \\
\underline{-abx+b^2} &
\end{array}$$

21. Arranging the dividend according to the powers of the letters as found in the divisor:

$$\begin{array}{l|l}
-18b^3+27ab^2+20a^2b-25a^3 & 6b-5a \\ \hline
-18b^3+15ab^2 & -3b^2+2ab+5a^2 \\ \hline
12ab^2+20a^2b & \\
\underline{12ab^2-10a^2b} & \\
\qquad 30a^2b-25a^3 & \\
\qquad \underline{30a^2b-25a^3} &
\end{array}$$

22. Arranging the dividend:

$$\begin{array}{l|l}
3x^4-8x^2y^2+5y^4+3x^2z^2-3y^2z^2 & x^2-y^2 \\ \hline
3x^4-3x^2y^2 & 3x^2-5y^2+3z^2 \\ \hline
-5x^2y^2+5y^4 & \\
\underline{-5x^2y^2+5y^4} & \\
\qquad 3x^2z^2-3y^2z^2 & \\
\qquad \underline{3x^2z^2-3y^2z^2} &
\end{array}$$

23.
$$\begin{array}{l|l}
4a^4-9a^2+6a-1 & 2a^2+3a-1 \\ \hline
4a^4+6a^3-2a^2 & 2a^2-3a+1 \\ \hline
-6a^3-7a^2+6a-1 & \\
\underline{-6a^3-9a^2+3a} & \\
\qquad 2a^2+3a-1 & \\
\qquad \underline{2a^2+3a-1} &
\end{array}$$

25. The dividend may be arranged, or we may divide the highest powers in the dividend by the divisor, without arranging the terms:

$$\begin{array}{l|l}
b - 6b^3 - 2a + 54a^3 - 3a^2b & 2a - b \\ \cline{2-2}
\underline{\qquad\qquad\qquad + 54a^3 - 27a^2b} & 27a^2 + 12ab + 6b^2 - 1 \\
b - 6b^3 - 2a + 24a^2b & \\
\underline{\qquad\quad - 12ab^2 + 24a^2b} & \\
b - 6b^3 - 2a + 12ab^2 & \\
\underline{\quad - 6b^3 \qquad\quad + 12ab^2} & \\
b - 2a & \\
\underline{b - 2a} &
\end{array}$$

26. Arranging the dividend:

$$\begin{array}{l|l}
25a^5 - a^3 - 2a^2 - 8a & 5a^2 + 4a \\ \cline{2-2}
\underline{25a^5 - 20a^4 \qquad\qquad\quad} & 5a^3 + 4a^2 + 3a + 2 \\
20a^4 - a^3 & \\
\underline{20a^4 - 16a^3} & \\
\qquad 15a^3 - 2a^2 & \\
\qquad \underline{15a^3 - 12a^2} & \\
\qquad\qquad 10a^2 - 8a & \\
\qquad\qquad \underline{10a^2 - 8a} &
\end{array}$$

27.

$$\begin{array}{l|l}
x^3 + y^3 + z^3 - 3xyz & x + y + z \\ \cline{2-2}
\underline{x^3 + x^2y + x^2z} & x^2 - xy + y^2 - xz - yz + z^2 \\
-x^2y + y^3 + z^3 - x^2z - 3xyz & \\
\underline{-x^2y - xy^2 \qquad\qquad - xyz} & \\
\qquad xy^2 + y^3 + z^3 - x^2z - 2xyz & \\
\qquad \underline{xy^2 + y^3 + y^2z} & \\
\qquad\qquad -x^2z - 2xyz - y^2z + z^3 & \\
\qquad\qquad \underline{-x^2z - xyz - xz^2} & \\
\qquad\qquad\qquad -xyz + xz^2 - y^2z + z^3 & \\
\qquad\qquad\qquad \underline{-xyz \qquad - y^2z - yz^2} & \\
\qquad\qquad\qquad\qquad xz^2 + yz^2 + z^3 & \\
\qquad\qquad\qquad\qquad \underline{xz^2 + yz^2 + z^3} &
\end{array}$$

28.

$$\begin{array}{l|l}
18x^4 - 45x^3 + 82x^2 - 67x + 40 & 3x^2 - 4x + 5 \\ \cline{2-2}
\underline{18x^4 - 24x^3 + 30x^2 \qquad\qquad} & 6x^2 - 7x + 8 \\
-21x^3 + 52x^2 - 67x & \\
\underline{-21x^3 + 28x^2 - 35x} & \\
\qquad 24x^2 - 32x + 40 & \\
\qquad \underline{24x^2 - 32x + 40} &
\end{array}$$

Page 64.

29.

$$\begin{array}{l|l}
16x^4 - 72a^2x^2 + 81a^4 & 2x - 3a \\ \cline{2-2}
16x^4 - 24ax^3 & 8x^3 + 12ax^2 - 18a^2x - 27a^3 \\ \cline{1-1}
24ax^3 - 72a^2x^2 & \\
24ax^3 - 36a^2x^2 & \\ \cline{1-1}
\quad - 36a^2x^2 + 81a^4 & \\
\quad - 36a^2x^2 + 54a^3x & \\ \cline{1-1}
\qquad - 54a^3x + 81a^4 & \\
\qquad - 54a^3x + 81a^4 & \\ \cline{1-1}
\end{array}$$

30.

$$\begin{array}{l|l}
a^4 + 4a^2x^2 + 16x^4 & a^2 + 2ax + 4x^2 \\ \cline{2-2}
a^4 + 4a^2x^2 + 2a^3x & a^2 - 2ax + 4x^2 \\ \cline{1-2}
\quad - 2a^3x + 16x^4 & \\
\quad - 2a^3x - 4a^2x^2 - 8ax^3 & \\ \cline{1-1}
\qquad 4a^2x^2 + 8ax^3 + 16x^4 & \\
\qquad 4a^2x^2 + 8ax^3 + 16x^4 & \\ \cline{1-1}
\end{array}$$

31.

$$\begin{array}{l|l}
x^4 + x^2z^2 + z^4 & x^2 - xz + z^2 \\ \cline{2-2}
x^4 + x^2z^2 - x^3z & x^2 + xz + z^2 \\ \cline{1-2}
\quad x^3z + z^4 & \\
\quad x^3z - x^2z^2 + xz^3 & \\ \cline{1-1}
\qquad x^2z^2 - xz^3 + z^4 & \\
\qquad x^2z^2 - xz^3 + z^4 & \\ \cline{1-1}
\end{array}$$

32.

$$\begin{array}{l|l}
x^4 - y^4 & x - y \\ \cline{2-2}
x^4 - x^3y & x^3 + x^2y + xy^2 + y^3 \\ \cline{1-2}
\quad x^3y - y^4 & \\
\quad x^3y - x^2y^2 & \\ \cline{1-1}
\qquad x^2y^2 - y^4 & \\
\qquad x^2y^2 - xy^3 & \\ \cline{1-1}
\qquad\quad xy^3 - y^4 & \\
\qquad\quad xy^3 - y^4 & \\ \cline{1-1}
\end{array}$$

33.

$$\begin{array}{l|l}
x^5 + y^5 & x + y \\ \cline{2-2}
x^5 + x^4y & x^4 - x^3y + x^2y^2 - xy^3 + y^4 \\ \cline{1-2}
\quad - x^4y + y^5 & \\
\quad - x^4y - x^3y^2 & \\ \cline{1-1}
\qquad x^3y^2 + y^5 & \\
\qquad x^3y^2 + x^2y^3 & \\ \cline{1-1}
\qquad\quad - x^2y^3 + y^5 & \\
\qquad\quad - x^2y^3 - xy^4 & \\ \cline{1-1}
\qquad\qquad xy^4 + y^5 & \\
\qquad\qquad xy^4 + y^5 & \\ \cline{1-1}
\end{array}$$

34.

$$\begin{array}{l|l}
x^7 + 1 & x + 1 \\ \cline{2-2}
x^7 + x^6 & x^6 - x^5 + x^4 - x^3 + x^2 - x + 1 \\ \cline{1-2}
\quad - x^3 + 1 & \\
\quad - x^6 - x^5 & \\ \cline{1-1}
\qquad x^5 + 1 & \\
\qquad x^5 + x^4 & \\ \cline{1-1}
\qquad\quad - x^4 + 1 & \\
\qquad\quad - x^4 - x^3 & \\ \cline{1-1}
\qquad\qquad x^3 + 1 & \\
\qquad\qquad x^3 + x^2 & \\ \cline{1-1}
\qquad\qquad\quad - x^2 + 1 & \\
\qquad\qquad\quad - x^2 - x & \\ \cline{1-1}
\qquad\qquad\qquad x + 1 & \\
\qquad\qquad\qquad x + 1 & \\ \cline{1-1}
\end{array}$$

35.

$$\begin{array}{l|l}
x^4 - 81y^4 & x - 3y \\ \cline{2-2}
x^4 - 3x^3y & x^3 + 3x^2y + 9xy^2 + 27y^3 \\ \cline{1-2}
\quad 3x^3y - 81y^4 & \\
\quad 3x^3y - 9x^2y^2 & \\ \cline{1-1}
\qquad 9x^2y^2 - 81y^4 & \\
\qquad 9x^2y^2 - 27xy^3 & \\ \cline{1-1}
\qquad\quad 27xy^3 - 81y^4 & \\
\qquad\quad 27xy^3 - 81y^4 & \\ \cline{1-1}
\end{array}$$

36.
$$\begin{array}{l|l}
81a^4 - 16b^4 & 3a + 2b \\
\underline{81a^4 + 54a^3b} & \overline{27a^3 - 18a^2b + 12ab^2 - 8b^3} \\
\quad - 54a^3b - 16b^4 & \\
\quad \underline{- 54a^3b - 36a^2b^2} & \\
\qquad\qquad 36a^2b^2 - 16b^4 & \\
\qquad\qquad \underline{36a^2b^2 + 24ab^3} & \\
\qquad\qquad\qquad - 24ab^3 - 16b^4 & \\
\qquad\qquad\qquad \underline{- 24ab^3 - 16b^4} &
\end{array}$$

37.
$$\begin{array}{l|l}
x^n - y^n & x + y \\
\underline{x^n + x^{n-1}y} & \overline{x^{n-1} - x^{n-2}y + x^{n-3}y^2} \\
\quad - x^{n-1}y - y^n & \\
\quad \underline{- x^{n-1}y - x^{n-2}y^2} & \\
\qquad\qquad x^{n-2}y^2 - y^n & \\
\qquad\qquad \underline{x^{n-2}y^2 + x^{n-3}y^3} & \\
\qquad\qquad\qquad - x^{n-3}y^3 - y^n &
\end{array}$$

38.
$$\begin{array}{l|l}
a^4 + 2a^2b^2 + 9b^4 & a^2 - 2ab + 3b^2 \\
\underline{a^4 + 3a^2b^2 - 2a^3b} & \overline{a^2 + 2ab + 3b^2} \\
2a^3b - a^2b^2 + 9b^4 & \\
\underline{2a^3b - 4a^2b^2 + 6ab^3} & \\
\qquad 3a^2b^2 - 6ab^3 + 9b^4 & \\
\qquad \underline{3a^2b^2 - 6ab^3 + 9b^4} &
\end{array}$$

39.
$$\begin{array}{l|l}
8x^3 + 27y^3 & 2x + 3y \\
\underline{8x^3 + 12x^2y} & \overline{4x^2 - 6xy + 9y^2} \\
- 12x^2y + 27y^3 & \\
\underline{- 12x^2y - 18xy^2} & \\
\qquad\qquad 18xy^2 + 27y^3 & \\
\qquad\qquad \underline{18xy^2 + 27y^3} &
\end{array}$$

40.
$$\begin{array}{l|l}
x^4 - a^2x^2 + 2a^3x - a^4 & x^2 - ax + a^2 \\
\underline{x^4 + a^2x^2 - ax^3} & \overline{x^2 + ax - a^2} \\
ax^3 - 2a^2x^2 + 2a^3x - a^4 & \\
\underline{ax^3 - a^2x^2 + a^3x} & \\
\quad - a^2x^2 + a^3x - a^4 & \\
\quad \underline{- a^2x^2 + a^3x - a^4} &
\end{array}$$

41.
$$\begin{array}{l|l}
a^4 - 4a^3x + 6a^2x^2 - 4ax^3 + x^4 & a^2 - 2ax + x^2 \\
\underline{a^4 - 2a^3x + a^2x^2} & \overline{a^2 - 2ax + x^2} \\
- 2a^3x + 5a^2x^2 - 4ax^3 + x^4 & \\
\underline{- 2a^3x + 4a^2x^2 - 2ax^3} & \\
\qquad\qquad a^2x^2 - 2ax^3 + x^4 & \\
\qquad\qquad \underline{a^2x^2 - 2ax^3 + x^4} &
\end{array}$$

42.
$$\begin{array}{l|l}
x^4 + 11x^2 - 12x - 5x^3 + 6 & x^2 - 3x + 3 \\
\underline{x^4 + 3x^2 \qquad - 3x^3} & \overline{x^2 - 2x + 2} \\
- 2x^3 + 8x^2 - 12x + 6 & \\
\underline{- 2x^3 + 6x^2 - 6x} & \\
\qquad\quad 2x^2 - 6x + 6 & \\
\qquad\quad \underline{2x^2 - 6x + 6} &
\end{array}$$

43.

$$\begin{array}{l|l}
8m^9 - 27n^6 & 2m^3 - 3n^2 \\ \cline{2-2}
\underline{8m^9 - 12m^6n^2} & 4m^6 + 6m^3n^2 + 9n^4 \\
12m^6n^2 - 27n^6 & \\
\underline{12m^6n^2 - 18m^3n^4} & \\
\qquad\qquad 18m^3n^4 - 27n^6 & \\
\qquad\qquad \underline{18m^3n^4 - 27n^6} &
\end{array}$$

44. Arranging the dividend:

$$\begin{array}{l|l}
x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4 & x - y \\ \cline{2-2}
\underline{x^4 - x^3y} & x^3 - 3x^2y + 3xy^2 - y^3 \\
-3x^3y + 6x^2y^2 & \\
\underline{-3x^3y + 3x^2y^2} & \\
\qquad\qquad 3x^2y^2 - 4xy^3 + y^4 & \\
\qquad\qquad \underline{3x^2y^2 - 3xy^3} & \\
\qquad\qquad\qquad\quad -\ xy^3 + y^4 & \\
\qquad\qquad\qquad\quad \underline{-\ xy^3 + y^4} &
\end{array}$$

45. Arranging the dividend:

$$\begin{array}{l|l}
x^6 - x^4 + x^3 - x^2 + 2x - 1 & x^2 + x - 1 \\ \cline{2-2}
\underline{x^6 - x^4 + x^5} & x^4 - x^3 + x^2 - x + 1 \\
-x^5 + x^3 - x^2 & \\
\underline{-x^5 + x^3 - x^4} & \\
\qquad\quad +x^4 - x^2 + 2x - 1 & \\
\qquad\quad\ \underline{x^4 - x^2 + x^3} & \\
\qquad\qquad\quad -x^3 + 2x - 1 & \\
\qquad\qquad\quad \underline{-x^3 + \ \ x - x^2} & \\
\qquad\qquad\qquad\qquad x^2 + x - 1 & \\
\qquad\qquad\qquad\qquad \underline{x^2 + x - 1} &
\end{array}$$

46. Expanding the divisor:

$$\begin{array}{l|l}
a^5 - b^5 & a^4 + a^3b + a^2b^2 + ab^3 + b^4 \\ \cline{2-2}
\underline{a^5 + a^4b + a^3b^2 + a^2b^3 + ab^4} & a - b \\
-a^4b - a^3b^2 - a^2b^3 - ab^4 - b^5 & \\
\underline{-a^4b - a^3b^2 - a^2b^3 - ab^4 - b^5} &
\end{array}$$

47.

$$\begin{array}{l|l}
a^6 - b^6 & a^3 - 2a^2b + 2ab^2 - b^3 \\ \cline{2-2}
\underline{a^6 - 2a^5b + 2a^4b^2 - a^3b^3} & a^3 + 2a^2b + 2ab^2 + b^3 \\
2a^5b - 2a^4b^2 + a^3b^3 - b^6 & \\
\underline{2a^5b - 4a^4b^2 + 4a^3b^3 - 2a^2b^4} & \\
\qquad\quad 2a^4b^2 - 3a^3b^3 + 2a^2b^4 - b^6 & \\
\qquad\quad \underline{2a^4b^2 - 4a^3b^3 + 4a^2b^4 - 2ab^5} & \\
\qquad\qquad\qquad a^3b^3 - 2a^2b^4 + 2ab^5 - b^6 & \\
\qquad\qquad\qquad \underline{a^3b^3 - 2a^2b^4 + 2ab^5 - b^6} &
\end{array}$$

DIVISION — EQUATIONS AND PROBLEMS.

Page 65.

3. $cx - 9 = c^2 + 6c - 3x;$ (1)

transposing, $cx + 3x = c^2 + 6c + 9;$ (2)

$(c + 3)x = c^2 + 6c + 9.$ (3)

Dividing by $(c + 3)$, $x = c + 3.$

4. $ax + 16 = a^2 - 4x;$ (1)

transposing, $ax + 4x = a^2 - 16;$ (2)

$(a + 4)x = a^2 - 16.$ (3)

Dividing by $(a + 4)$, $x = a - 4.$

5. $3x - 12a = 4a^2 - 2ax + 9;$ (1)

transposing, $3x + 3ax = 4a^2 + 12a + 9;$ (2)

$(3 + 2a)x = 4a^2 + 12a + 9.$ (3)

Dividing by $(3 + 2a)$, $x = 2a + 3.$

6. $dx + 9a^2 = d^2 - 3ax;$ (1)

transposing, $dx + 3ax = d^2 - 9a^2;$ (2)

$(d + 3a)x = d^2 - 9a^2.$ (3)

Dividing by $(d + 3a)$, $x = d - 3a.$

Page 66.

7. $ax - a^2 = 2ab + b^2 - bx;$ (1)

transposing, $ax + bx = a^2 + 2ab + b^2;$ (2)

$(a + b)x = a^2 + 2ab + b^2.$ (3)

Dividing by $(a + b)$, $x = a + b.$

8. $ax - 5ab = 2a^2 + 3b^2 - bx;$ (1)

transposing, $ax + bx = 2a^2 + 5ab + 3b^2;$ (2)

$(a + b)x = 2a^2 + 5ab + 3b^2.$ (3)

Dividing by $(a + b)$, $x = 2a + 3b.$

9. $ax - c^2 = a^3 + ac + a^2c - cx;$ (1)

transposing, $ax + cx = a^3 + ac + a^2c + c^2;$ (2)

$(a + c)x = a^3 + ac + a^2c + c^2.$ (3)

Dividing by $(a + c)$, $x = a^2 + c.$

10. $2ax - 6a^2 = 13ab + 6b^2 - 3bx;$ (1)

transposing, $2ax + 3bx = 6a^2 + 13ab + 6b^2;$ (2)

$(2a + 3b)x = 6a^2 + 13ab + 6b^2.$ (3)

Dividing by $(2a + 3b)$, $x = 3a + 2b.$

11. $2ax - 10ab - 15b = 14a + 21 - 3x;$ (1)

transposing, $2ax + 3x = 10ab + 15b + 14a + 21;$ (2)

$(2a + 3)x = 10ab + 15b + 14a + 21.$ (3)

Dividing by $(2a + 3)$, $x = 5b + 7.$

12. $ax + bx = 5a^2 + 7ab + 2b^2 + 5ac + 2bc - cx;$ (1)

transposing, $ax + bx + cx = 5a^2 + 7ab + 2b^2 + 5ac + 2bc;$ (2)

$(a + b + c)x = 5a^2 + 7ab + 2b^2 + 5ac + 2bc.$ (3)

Dividing by $(a + b + c)$, $x = 5a + 2b.$

13. $2cx - 4c^3 + d^2 = 2c^2d - 2cd - dx;$ (1)

transposing, $2cx + dx = 2c^2d - 2cd + 4c^3 - d^2;$ (2)

$(2c + d)x = 2c^2d - 2cd + 4c^3 - d^2.$ (3)

Dividing by $(2c + d)$, $x = 2c^2 - d.$

14. $b^2x + 3b^2c + 6c^3 = b^3 + 2bc^2 - 2c^2x;$ (1)

transposing, $b^2x + 2c^2x = b^3 - 3b^2c + 2bc^2 - 6c^3;$ (2)

$(b^2 + 2c^2)x = b^3 - 3b^2c + 2bc^2 - 6c^3.$ (3)

Dividing by $(b^2 + 2c^2)$, $x = b - 3c.$

15. $4m^4 - 2m^2x - 3mx = 1 - 6m + 9m^2 - x;$ (1)

transposing, $x - 2m^2x - 3mx = 1 - 6m + 9m^2 - 4m^4;$ (2)

$(1 - 2m^2 - 3m)x = 1 - 6m + 9m^2 - 4m^4.$ (3)

Dividing by $(1 - 2m^2 - 3m)$, $x = 1 - 3m + 2m^2.$

16. $a^3 + 3x - 9a^2 = ax - 27a + 27;$ (1)

transposing, $3x - ax = 27 - 27a + 9a^2 - a^3;$ (2)

$(3 - a)x = 27 - 27a + 9a^2 - a^3.$ (3)

Dividing by $(3 - a)$, $x = 9 - 6a + a^2.$

17. $2m^2x + 3mn^3 + 7m^2n^2 - 4m^4 = 3mnx;$ (1)

transposing, $2m^2x - 3mnx = 4m^4 - 3mn^3 - 7m^2n^2;$ (2)

$(2m^2 - 3mn)x = 4m^4 - 3mn^3 - 7m^2n^2.$ (3)

Dividing by $(2m^2 - 3mn)$, $x = 2m^2 + 3mn + n^2.$

18. $5ax = 15a^3 - 5ab + 5ab^2 + 2bx - 6a^2b + 2b^2 - 2b^3;$ (1)

transposing, $5ax - 2bx = 15a^3 - 6a^2b - 5ab + 5ab^2 + 2b^2 - 2b^3;$ (2)

$(5a - 2b)x = 15a^3 - 6a^2b - 5ab + 5ab^2 + 2b^2 - 2b^3.$ (3)

Dividing by $(5a - 2b)$, $x = 3a^2 - b + b^2.$

19. Let x represent the sum of money;

then, $3x + 25 = 355.$

Transposing, etc., $x = 110$ dollars, the sum of money.

20. Let x represent the sum given to first;

then, $2x =$ the sum given to second;

$4x =$ the sum given to third;

$3\frac{1}{2}x =$ the sum given to fourth.

Therefore, $x + 2x + 4x + 3\frac{1}{2}x = 10500;$

whence, $x = 1000$ dollars, the share of first;

$2x = 2000$ dollars, the share of second;

$4x = 4000$ dollars, the share of third;

$3\frac{1}{2}x = 3500$ dollars, the share of fourth.

21. Let x represent the number of beggars;

then, $3x + 4 = 5x - 6.$

Transposing, etc., $x = 5$, the number of beggars;

$3x + 4 = 19$, the number of cents.

22. Let x represent the age of the youngest;

then, $x + 4 =$ the age of the next older;

and, $x + 20 =$ the age of the oldest.

Therefore, $3x = x + 20;$

whence, $x = 10$, the age of the youngest;

and 10, 14, 18, 22, 26, 30, are the ages.

23. Let x represent the quantity at first;

then, $x + 42 = 7x;$

whence, $x = 7$ gallons, the quantity at first.

Page 67.

24. Let x represent the amount he had at first;

then, $2x =$ the amount he had after borrowing first time;

$2x - 1 =$ the amount left after spending the first dollar;

$4x - 2 =$ the amount he had after borrowing second time;

$4x - 3 =$ the amount left after spending the second dollar;

$8x - 6 =$ the amount he had after borrowing the third time,

$8x - 7 =$ the amount left after spending the third dollar.

Therefore, $8x - 7 = 0$;

whence, $x = \frac{7}{8}$ dollars, the sum he had at first.

25. Let $x =$ B's share;

$x + 100 =$ A's share;

$x + 300 =$ C's share;

then, $3x + 400 = 3400$;

$3x = 3000$.

Whence, $x = 1000$ dollars, B's;

$x + 100 = 1100$ dollars, A's;

$x + 300 = 1300$ dollars, C's.

26. Let $x =$ num. in 1st flock;

$360 - x =$ num. in 2d flock;

then, $\frac{x + 40}{2} = 320 - x$;

$x + 40 = 640 - 2x$;

$3x = 600$.

Whence, $x = 200$ sheep, in 1st;

$360 - x = 160$ sheep, in 2d.

27. Let $x =$ the sum;

then, $x + 252 = 2(x - 174)$.

Whence, $x = 600$ dollars.

28. Let $x =$ money of 2d;

$x - 20 =$ money of 1st;

$2x - 20 =$ money of 3d;

then, $4x - 40 = 152$,

$4x = 192$.

Whence, $x = 48$ dollars, 2d;

$x - 20 = 28$ dollars, 1st;

$2x - 20 = 76$ dollars, 3d.

29. Let $x =$ 1st part;

$x + 3 =$ 2d part;

$x - 2 =$ 3d part;

then, $3x + 1 = 37$;

$3x = 36$.

Whence, $x = 12$, 1st;

$x + 3 = 15$, 2d;

$x - 2 = 10$, 3d.

30. Let $x =$ num. to Denver;

$x + 10 =$ num. to Omaha;

$3x + 35 =$ num. to St. Paul;

$6x + 70 =$ num. to Chicago;

then, $11x + 115 = 1215$.

Whence, $x = 100$ to D.;

$x + 10 = 110$ to O.;

$3x + 35 = 335$ to St. P.:

$6x + 70 = 670$ to C.

REVIEW.

Page 68.

33. $$(1+x)(1-x)=1-x^2.$$
$$(1-x^2)(1+x^2)=1-x^4.$$
$$(1-x^4)(1+x^4)=1-x^8.$$
$$(1-x^8)(1+x^8)=1-x^{16}.$$

35. $$(a+2)(a-2)=a^2-4. \qquad (1)$$
$$(a+2)(a-2)=a^2-4. \qquad (2)$$
$(1)\times(2)$, $$(a^2-4)(a^2-4)=a^4-8\,a^2+16.$$

36. $$(3\,a-6)(3\,a+6)=9\,a^2-36. \qquad (1)$$
$$(3\,a-6)(3\,a+6)=9\,a^2-36. \qquad (2)$$
$(1)\times(2)$, $$(9\,a^2-36)(9\,a^2-36)=81\,a^4-648\,a^2+1296.$$

Page 69.

63. $$(a+x)(a-x)=a^2-x^2.$$
$$(a^2-x^2)(a^2+x^2)=a^4-x^4.$$
$$(a^4-x^4)(a^4+x^4)=a^8-x^8.$$

64. $$(x^2+y^2)(x^2-y^2)=x^4-y^4.$$
$$(x^4-y^4)(x^4+y^4)=x^8-y^8.$$
$$(x^8-y^8)(x^8+y^8)=x^{16}-y^{16}.$$

65. $$(2\,x+3)(2\,x-3)=4\,x^2-9.$$
$$(4\,x^2-9)(4\,x^2+9)=16\,x^4-81.$$
$$(16\,x^4-81)(16\,x^4-81)=256\,x^8-2592\,x^4+6561.$$

Page 70.

76. $$(a^2-\overline{2\,ax-x^2})-\{a^2-(2\,ax-\overline{x^2+7})\}$$
$$=a^2-2\,ax+x^2-\{a^2-2\,ax+\overline{x^2+7}\}$$
$$=a^2-2\,ax+x^2-a^2+2\,ax-x^2-7=-7.$$

77. $$a^3-[(-a^3-a^2b+ab^2-b^3)-\{a^2b-(ab^2-\overline{b^3-a^2b})\}]$$
$$=a^3-[-a^3-a^2b+ab^2-b^3-\{a^2b-ab^2+\overline{b^3-a^2b}\}]$$
$$=a^3-[-a^3-a^2b+ab^2-b^3-a^2b+ab^2-b^3+a^2b]$$
$$=a^3+a^3+a^2b-ab^2+b^3+a^2b-ab^2+b^3-a^2b$$
$$=2\,a^3+a^2b-2\,ab^2+2\,b^3.$$

78. $$\begin{aligned}&4a^2-(a^2-b^2)-[-\{b^2-(a^2-\overline{b^2-a^2})\}]\\&=4a^2-a^2+b^2-[-\{b^2-a^2+b^2-a^2\}]\\&=4a^2-a^2+b^2+\{b^2-a^2+b^2-a^2\}\\&=4a^2-a^2+b^2+b^2-a^2+b^2-a^2=a^2+3b^2.\end{aligned}$$

79. $$\begin{aligned}&1-[4-\{7-(2-\{3-\overline{4-a}\})\}]=1-[4-\{7-(2-3+4-a)\}]\\&=1-4+\{7-(2-3+4-a)\}=1-4+7-2+3-4+a=1+a.\end{aligned}$$

80. $$\begin{aligned}&14-\{-7+(3-\overline{4+x})-x\}-[-\{-(-\overline{x-4})\}]\\&=14-\{-7+3-4-x-x\}+\{-(-\overline{x-4})\}\\&=14+7-3+4+x+x+\{\overline{x-4}\}\\&=14+7-3+4+x+x+x-4=3x+18.\end{aligned}$$

81. $$\begin{aligned}&a-[2b-(3c+2b)-a]=a-[2b-3c-2b-a]\\&=a-2b+3c+2b+a=2a+3c.\end{aligned}$$

82. $$\begin{aligned}&a-\{2a-[3b-(4c-2a)]\}=a-2a+[3b-(4c-2a)]\\&=a-2a+3b-4c+2a=a+3b-4c.\end{aligned}$$

83. $$\begin{aligned}&7x-[3x-(4x-\overline{5x-2x})]=7x-3x+(4x-\overline{5x-2x})\\&=7x-3x+4x-5x+2x=5x.\end{aligned}$$

84. $$\begin{aligned}&-[5a-(11b-3a)]-[5b-(3a-6b)]\\&=-[5a-11b+3a]-[5b-3a+6b]\\&=-5a+11b-3a-5b+3a-6b=-5a.\end{aligned}$$

85. $$\begin{aligned}&x-\{2x+3x-[4x-5x-(6x-7x)]\}\\&=x-\{2x+3x-[4x-5x-6x+7x]\}\\&=x-\{2x+3x-4x+5x+6x-7x\}\\&=x-2x-3x+4x-5x-6x+7x=-4x.\end{aligned}$$

86. $$\begin{aligned}&8x-\{16y-[3x-(12y-x)-8y]+x\}\\&=8x-\{16y-[3x-12y+x-8y]+x\}\\&=8x-\{16y-3x+12y-x+8y+x\}\\&=8x-16y+3x-12y+x-8y-x=11x-36y.\end{aligned}$$

87. $$\begin{aligned}&a-2b+\{3c-[3a-(a+b+2a-b)]\}\\&=a-2b+3c-[3a-a-b-2a+b]\\&=a-2b+3c-3a+a+b+2a-b=a-2b+3c.\end{aligned}$$

88. $2x - \{2x - [2x - (2x - \overline{2x - x})]\}$
$= 2x - \{2x - [2x - 2x + 2x - x]\}$
$= 2x - \{2x - 2x + 2x - 2x + x\}$
$= 2x - 2x + 2x - 2x + 2x - x = x.$

89. $a - \{2b - [3c - 2a - (a + b + 2a)]\}$
$= a - \{2b - [3c - 2a - a - b - 2a]\}$
$= a - \{2b - 3c + 2a + a + b + 2a\}$
$= a - 2b + 3c - 2a - a - b - 2a = 3c - 4a - 3b.$

90. $2x - \{3y + (2y - z) - 4z + [2x - (3y - \overline{z - 2y})]\}$
$= 2x - \{3y + 2y - z - 4z + 2x - (3y - \overline{z - 2y})\}$
$= 2x - \{3y + 2y - z - 4z + 2x - 3y + z - 2y\}$
$= 2x - 3y - 2y + z + 4z - 2x + 3y - z + 2y = 4z.$

Page 71.

119. $7x - 36 - 10x + 12 = 80 - 33x + 46;$
transposing, etc., $30x = 150.$
Whence, $x = 5.$

120. $6x - 18 + 8x + 15x - 21 = 10x - 4 - 16x$
transposing, etc., $35x = 35$
Whence, $x = 1.$

121. $3x - 6x + 30 + 2x = 2x + 10 + 5x - 20;$
transposing, etc., $-8x = -40.$
Whence, $x = 5.$

122. $x - 28x + 77 + 30 = 14x - 70 - 152 + 19x - 31;$
transposing, etc., $60x = 360.$
Whence, $x = 6.$

123. $17x - 8x + 9 - 4 + 3x + 2x - 3 = 30;$
transposing, etc., $14x = 28.$
Whence, $x = 2.$

124. $x^2 + 4x - 96 = x^2 - 5x - 6;$
cancelling x^2 and transposing, etc., $9x = 90.$
Whence, $x = 10.$

125. $4x-12-3x+6+2x-2+2=0.$

Whence, $3x=6;$

and, $x=2.$

126. $2x^2-x-1+14=2x^2+9x+9;$

cancelling $2x^2$'s, and trans., $10x=4.$

Whence, $x=\frac{2}{5}.$

127. $x^3+28x-4=-9x^2+x^3+9x^2+20x+12;$

cancelling x^3's and x^2's, $8x=16.$

Whence, $x=2.$

128. $ax-a^2-2ab=-bx+b^2.$

Whence, $(a+b)x=(a+b)^2;$

and, $x=a+b.$

129. $(a^2+b^2)x+x^2+a^2b^2=a^2b^2+2abx+2x^2;$

cancelling a^2b^2, and trans., $(a-b)^2x=x^2.$

Dividing each member by x, $x=a^2-2ab+b^2.$

130. $(a-b)x=a^2-b^2.$

Whence, $x=a+b.$

131. $(b-3c)x=7bc-3b^2.$

Whence, $x=\dfrac{7bc-3b^2}{b-3c}.$

132. $a(x+a)+b(b-x)=2ab;$

$ax+a^2+b^2-bx=2ab;$

transposing, etc., $(a-b)x=-(a-b)^2.$

Whence, $x=-(a-b)$; or, $b-a.$

133. $2ax+12ab-4a^2=9b^2+3bx;$

transposing, $2ax-3bx=4a^2-12ab+9b^2;$

$(2a-3b)x=4a^2-12ab+9b^2.$

Dividing by $(2a-3b)$, $x=2a-3b.$

134. $3x-9-3c=12a-2ax+4a^2+2ac;$

transposing, $3x+2ax=9+12a+4a^2+3c+2ac;$

$(3+2a)x=9+12a+4a^2+3c+2ac.$

Dividing by $(3+2a)$, $x=3+2a+c.$

135. $$2ax + 9c^2 + 3cd = 4a^2 + 3cx + 2ad;$$

transposing, $$2ax - 3cx = 4a^2 - 9c^2 + 2ad - 3cd;$$

$$(2a - 3c)x = 4a^2 - 9c^2 + 2ad - 3cd.$$

Dividing by $(2a - 3c)$, $$x = 2a + 3c + d.$$

FACTORING.

Page 78.

59. $b^2 + 2bc + c^2 - d^2 = (b + c)^2 - d^2 = (b + c + d)(b + c - d).$

60. $a^2 + 2ax + x^2 - 4y^2 = (a + x)^2 - 4y^2 = (a + x + 2y)(a + x - 2y).$

61. $4a^2 - x^2 + 2xy - y^2 = 4a^2 - (x - y)^2 = (2a + x - y)(2a - x + y).$

62. $a^2 - 6ax + 9x^2 - 4c^2 = (a - 3x)^2 - 4c^2$
$= (a - 3x + 2c)(a - 3x - 2c).$

63. $4x^2 + 4xy + y^2 - 4z^2 = (2x + y)^2 - 4z^2$
$= (2x + y + 2z)(2x + y - 2z).$

64. $x^2 - b^2 - 2xy + y^2 = (x - y)^2 - b^2 = (x - y + b)(x - y - b).$

65. $x^2 + 4xy - 4z^2 + 4y^2 = (x + 2y)^2 - 4z^2$
$= (x + 2y + 2z)(x + 2y - 2z).$

66. $x^2 - 9y^2 + 6x + 9 = (x + 3)^2 - 9y^2 = (x + 3 + 3y)(x + 3 - 3y).$

67. $4x^2 + 12xy - 9c^2 + 9y^2 = (2x + 3y)^2 - 9c^2$
$= (2x + 3y + 3c)(2x + 3y - 3c).$

68. $4a^2 + 9b^2 - 16c^2 - 12ab = (2a - 3b)^2 - 16c^2$
$= (2a - 3b + 4c)(2a - 3b - 4c).$

69. $9x^2 - 25z^2 + 16y^2 + 24xy = (3x + 4y)^2 - 25z^2$
$= (3x + 4y + 5z)(3x + 4y - 5z).$

70. $a^2 + 2bd + c^2 - b^2 + 2ac - d^2 = (a + c)^2 - (b - d)^2$
$= (a + c + b - d)(a + c - b + d).$

71. $4b^2 - x^2 + 4xy + a^2 + 4ab - 4y^2 = (a + 2b)^2 - (x - 2y)^2$
$= (a + 2b + x - 2y)(a + 2b - x + 2y).$

72. $d^2 - a^2 + 4ax - 4cd + 4c^2 - 4x^2 = (d - 2c)^2 - (a - 2x)^2$
$= (d - 2c + a - 2x)(d - 2c - a + 2x).$

Page 82.

3. $b^4 + b^2c^2 + c^4 = b^4 + 2b^2c^2 + c^4 - b^2c^2 = (b^2 + c^2)^2 - b^2c^2$
$= (b^2 + c^2 + bc)(b^2 + c^2 - bc).$

4. $x^4 + x^2 + 1 = x^4 + 2x^2 + 1 - x^2 = (x^2 + 1)^2 - x^2$
$= (x^2 + 1 + x)(x^2 + 1 - x).$

5. $9b^4 + 21b^2c^2 + 25c^4 = 9bx + 30b^2c^2 + 25c^4 - 9b^2c^2$
$= (3b^2 + 5c^2)^2 - 9b^2c^2 = (3b^2 + 5c^2 + 3bc)(3b^2 + 5c^2 - 3bc).$

6. $9b^4 + 3b^2x^2 + 4x^4 = 9b^4 + 12b^2x^2 + 4x^4 - 9b^2x^2$
$= (3b^2 + 2x^2)^2 - 9b^2x^2 = (3b^2 + 2x^2 + 3bx)(3b^2 + 2x^2 - 3bx).$

7. $25x^4 - 9x^2y^2 + 16y^4 = 25x^4 + 40x^2y^2 + 16y^4 - 49x^2y^2$
$= (5x^2 + 4y^2)^2 - 49x^2y^2 = (5x^2 + 4y^2 - 7xy)(5x^2 + 4y^2 + 7xy).$

8. $16a^4 - 17a^2b^2 + b^4 = 16a^4 + 8a^2b^2 + b^4 - 25a^2b^2$
$= (4a^2 + b^2)^2 - 25a^2b^2 = (4a^2 + b^2 + 5ab)(4a^2 + b^2 - 5ab).$

or, $16a^4 - 17a^2b^2 + b^4 = 16a^4 - 8a^2b^2 + b^4 - 9a^2b^2$
$= (4a^2 - b^2)^2 - 9a^2b^2 = (4a^2 - b^2 + 3ab)(4a^2 - b^2 - 3ab).$

9. $36x^4 + 23x^2y^2 + 16y^4 = 36x^4 + 48x^2y^2 + 16y^4 - 25x^2y^2$
$= (6x^2 + 4y^2)^2 - 25x^2y^2 = (6x^2 + 4y^2 + 5xy)(6x^2 + 4y^2 - 5xy).$

10. $49b^4 - 11b^2c^2 + 25c^4 = 49b^4 + 70b^2c^2 + 25c^4 - 81b^2c^2$
$= (7b^2 + 5c^2)^2 - 81b^2c^2 = (7b^2 + 5c^2 + 9bc)(7b^2 + 5c^2 - 9bc).$

11. $64a^4 + 128a^2b^2 + 81b^4 = 64a^4 + 144a^2b^2 + 81b^4 - 16a^2b^2$
$= (8a^2 + 9b^2)^2 - 16a^2b^2 = (8a^2 + 9b^2 + 4ab)(8a^2 + 9b^2 - 4ab).$

12. $49x^4 + 34x^2y^2 + 25y^4 = 49x^4 + 70x^2y^2 + 25y^4 - 36x^2y^2$
$= (7x^2 + 5y^2)^2 - 36x^2y^2 = (7x^2 + 5y^2 + 6xy)(7x^2 + 5y^2 - 6xy).$

13. $9c^4 + 38c^2d^2 + 49d^4 = 9c^4 + 42c^2d^2 + 49d^4 - 4c^2d^2$
$= (3c^2 + 7d^2)^2 - 4c^2d^2 = (3c^2 + 7d^2 + 2cd)(3c^2 + 7d^2 - 2cd).$

14. $36a^4 - 16a^2b^2 + 49b^4 = 36a^4 + 84a^2b^2 + 49b^4 - 100a^2b^2$
$= (6a^2 + 7b^2)^2 - 100a^2b^2 = (6a^2 + 7b^2 + 10ab)(6a^2 + 7b^2 - 10ab).$

15. $25b^4 + 31b^2c^2 + 64c^4 = 25b^4 + 80b^2c^2 + 64c^4 - 49b^2c^2$
$= (5b^2 + 8c^2)^2 - 49b^2c^2 = (5b^2 + 8c^2 + 7bc)(5b^2 + 8c^2 - 7bc).$

16. $25b^4 - b^2c^2 + 64c^4 = 25b^4 + 80b^2c^2 + 64c^4 - 81b^2c^2$
$= (5b^2 + 8c^2)^2 - 81b^2c^2 = (5b^2 + 8c^2 + 9bc)(5b^2 + 8c^2 - 9bc).$

17. $81x^4 + 20x^2y^2 + 4y^4 = 81x^4 + 36x^2y^2 + 4y^4 - 16x^2y^2$
$= (9x^2 + 2y^2)^2 - 16x^2y^2 = (9x^2 + 2y^2 + 4xy)(9x^2 + 2y^2 - 4xy).$

18. $81x^4 - 64x^2y^2 + 4y^4 = 81x^4 + 36x^2y^2 + 4y^4 - 100x^2y^2$
$= (9x^2 + 2y^2)^2 - 100x^2y^2 = (9x^2 + 2y^2 + 10xy)(9x^2 + 2y^2 - 10xy).$

REVIEW IN FACTORING.

Page 87.

61. This is a perfect square, the middle term being twice the product of the square roots of the first and third; therefore, the factors are:

$$[(x^2 - xy) - (xy - 1)]\;[(x^2 - xy) - (xy - 1)],$$
$$= (x^2 - 2xy + 1)(x^2 - 2xy + 1).$$

62. See **61.** $[(3x - 4y) - (2x - 3y)]\;[(3x - 4y) - (2x - 3y)],$
$$= (x - y)(x - y).$$

63. This is the difference of two squares; therefore, the factors are.

$$[(a+b+c+d) - (a-b+c-d)]\;[(a+b+c+d) + (a-b+c-d)],$$
$$= (2b + 2d)(2a + 2c) = 4(b + d)(a + c).$$

64. The first term is the square of $(x - y + z)$; therefore, the factors are: $[(x - y + z) - (y + z)]\;[(x - y + z) + (y + z)],$
$$= (x - 2y)(x + 2z).$$

65. $x^2(y - z) - xy(y - z) = x(x - y)(y - z).$

66. $x^2(z^2 + a) - a^2(z^2 + a) = (x^2 - a^2)(z^2 + a) = (x - a)(x + a)(z^2 + a).$

67. $abxy + acx + b^2y^2 - c^2 = ax(by + c) + (by + c)(by - c)$
$= (by + c)(ax + by - c).$

68. $y^3 - 2ay^2 + a^2y - by^2 + 2aby - a^2b = y(y - a)^2 - b(y - a)^2$
$= (y - b)(y - a)(y - a).$

69. $-(m^2 - 2mn + n^2) + a^2 - 2ab + b^2 = (a - b)^2 - (m - n)^2$
$= (a - b + m - n)(a - b - m + n).$

70. $a^2 - 2ab + b^2 - (c^2 + 2cd + d^2) = (a - b)^2 - (c + d)^2$
$= (a - b + c + d)(a - b - c - d).$

71. $(a^2 - b^2)^2 - (m + n)^2 = (a^2 - b^2 + m + n)(a^2 - b^2 - m - n).$

72. $3a^2 - 6ab + 3b^2 + 6ac - 6bc = 3(a - b)^2 + 6c(a - b)$
$= (3a - 3b + 6c)(a - b) = 3(a - b + 2c)(a - b).$

73. The middle term is twice the product of the square roots of first and third; therefore, the factors are:

$$[(x^2 - 1) - (y^2 - 1)]\;[(x^2 - 1) - (y^2 - 1)],$$
$$= (x^2 - y^2)(x^2 - y^2) = (x + y)(x + y)(x - y)(x - y).$$

74. Factors are:

$$[(a^2+2ab+b^2)-(a^2-2ab+b^2)]\,[(a^2+2ab+b^2)+(a^2-2ab+b^2)],$$
$$=4ab \text{ times } 2(a^2+b^2)=8ab(a^2+b^2).$$

75. Factors are:

$$2a(x+y-z)+2b(x+y-z)=2(a+b)(x+y-z).$$

COMMON DIVISORS.

Page 90.

14.
$$a^2-b^2=(a+b)(a-b);$$
$$a^2-2ab+b^2=(a-b)(a-b).$$

Therefore, $(a-b)$ is the H. C. D.

15.
$$x^2-2x=x(x-2);$$
$$2xy^2-4y^2=2y^2(x-2).$$

Therefore, $x-2$ is the H. C. D.

16.
$$16x^2-y^2=(4x+y)(4x-y);$$
$$16x^2-8xy+y^2=(4x-y)(4x-y).$$

Therefore, $4x-y$ is the H. C. D.

17. $x^2-2x-15=(x+3)(x-5);$

$x^2+9x+18=(x+3)(x+6).$

Therefore, $x+3$ is the H. C. D.

18. $x^2+9x+20=(x+5)(x+4);$

$x^2+2x-15=(x+5)(x-3).$

Therefore, $x+5$ is the H. C. D.

19. $x^2+x-30=(x+6)(x-5);$

$x^2+12x+36=(x+6)(x+6).$

Therefore, $x+6$ is the H. C. D.

20. $x^2-x-12=(x-4)(x+3);$

$x^2-4x-21=(x-7)(x+3).$

Therefore, $x+3$ is the H. C. D.

21. $x^2+9x+14=(x+7)(x+2);$

$x^2+2x-35=(x+7)(x-5).$

Therefore, $x+7$ is the H. C. D.

22. $x^2+x-30=(x+6)(x-5);$

$x^2+9x+18=(x+6)(x+3).$

Therefore, $x+6$ is the H. C. D.

23.
$$a(x^4-y^4)=a(x^2+y^2)(x+y)(x-y);$$
$$x^4+2x^3y+x^2y^2=x^2(x+y)(x+y).$$

Therefore, $x+y$ is the H. C. D.

24.
$$4a^2-9x^2=(2a-3x)(2a+3x);$$
$$4a^2-12ax+9x^2=(2a-3x)(2a-3x).$$

Therefore, $2a-3x=$ H. C. D.

25. $b^4 - 27a^3b = b(b-3a)(b^2+3ab+9a^2);$

$(b-3a)^2 = (b-3a)(b-3a).$

Therefore, $b - 3a = \text{H. C. D.}$

26. $4ab^3 - 4abc^2 = 4a(b-c)(b+c)b;$

$12a^2b^2 - 12a^2c^2 = 4a(b-c)(b+c)3a.$

Therefore, $4a(b-c)(b+c) = \text{H. C. D.}$

27. $yz - z = z(y-1);$

$y^4z - yz = yz(y-1)(y^2+y+1).$

Therefore, $z(y-1) = \text{H. C. D.}$

28. $5a^2 - 2a - 3 = (5a+3)(a-1);$

$5a^2 - 11a + 6 = (5a-6)(a-1).$

Therefore, $a - 1 = \text{H. C. D.}$

29. $9y^2 - 4 = (3y-2)(3y+2);$

$9y^2 - 15y - 14 = (3y-7)(3y+2).$

Therefore, $3y + 2 = \text{H. C. D.}$

30. $(a^2 - b^2) = (a-b)(a+b);$

$a^2 - ab = (a-b)a;$

$(a-b)^2 = (a-b)(a-b).$

Therefore, $a - b = \text{H. C. D.}$

31. $8(x-y)^4 = 4 \times 2(x-y)(x-y)(x-y)(x-y);$

$10(x^2 - y^2) = 5 \times 2(x-y)(x+y);$

$12(x^4 - y^4) = 6 \times 2(x-y)(x+y)(x^2+y^2).$

Therefore, $2(x-y) = \text{H. C. D.}$

32. $a^2 - b^2 = (a-b)(a+b);$

$(a-b)^2 = (a-b)(a-b);$

$a^2b - ab^2 = (a-b)ab.$

Therefore, $a - b = \text{H. C. D.}$

33. $z^2 + z - 6 = (z+3)(z-2);$

$z^2 + 7z + 12 = (z+3)(z+4);$

$z^2 - 2z - 15 = (z+3)(z-5).$

Therefore, $z + 3 = \text{H. C. D.}$

34. $x^2 + 5x + 6 = (x+3)(x+2);$

$x^2 + 7x + 10 = (x+5)(x+2);$

$x^2 + 12x + 20 = (x+10)(x+2).$

Therefore, $x + 2 = \text{H. C. D.}$

35. $1 - a^2 = (1-a)(1+a);$

$1 + a^3 = (1-a+a^2)(1+a);$

$a^2 + 5a + 4 = (a+4)(a+1).$

Therefore, $a + 1 = \text{H. C. D.}$

Page 94.

4.

$$\begin{array}{r|l}
2x^2-16x+14 & x^2-5x-14 \\
\underline{2x^2-10x-28} & \underline{2\qquad\qquad\quad} \\
-6)\underline{-\ 6x+42} & \\
x^2-5x-14 & \underline{x-\ 7,\ \text{H. C. D.}} \\
\underline{x^2-7x\qquad} & \underline{x+\ 2\qquad\qquad} \\
2x-14 & \\
\underline{2x-14} &
\end{array}$$

5.

$$\begin{array}{r|l}
4x^2+19x+12 & 3x^2+14x+8 \\
\underline{3\qquad\qquad\quad} & \underline{4\qquad\qquad\quad} \\
12x^2+57x+36 & \\
\underline{12x^2+56x+32} & \\
3x^2+14x+8 & \underline{x+\ 4,\ \text{H. C. D.}} \\
\underline{3x^2+12x\quad} & \underline{3x+\ 2\qquad\qquad} \\
2x+8 & \\
\underline{2x+8} &
\end{array}$$

6.

$$\begin{array}{r|l}
6x^2-23x+15 & 2x^2-12x+18 \\
\underline{6x^2-36x+54} & \underline{3\qquad\qquad\qquad} \\
13)\underline{13x-39} & \\
2x^2-12x+18 & \underline{x-\ 3,\ \text{H. C. D.}} \\
\underline{2x^2-\ 6x\qquad} & \underline{2x-\ 6\qquad\qquad} \\
-\ 6x+18 & \\
\underline{-\ 6x+18} &
\end{array}$$

7.

$$\begin{array}{r|l}
4x^2+21x-18 & 2x^2+15x+18 \\
\underline{4x^2+30x+36} & \underline{2\qquad\qquad\qquad} \\
-9)\underline{-9x-54} & \\
2x^2+15x+18 & \underline{x+\ 6,\ \text{H. C. D.}} \\
\underline{2x^2+15x+18} & \underline{2x+\ 3\qquad\qquad}
\end{array}$$

8.

$$\begin{array}{r|l}
21x^2-26x+\ 8 & 6x^2-x-2 \\
\underline{2\qquad\qquad\qquad} & \underline{7\qquad\qquad} \\
42x^2-52x+16 & \\
\underline{42x^2-\ 7x-14} & \\
-15)\underline{-45x+30} & \\
6x^2-x-2 & \underline{3x-\ 2,\ \text{H. C. D.}} \\
\underline{6x^2-x-2} & \underline{2x+\ 1\qquad\qquad}
\end{array}$$

9.

$$\begin{array}{r|l}
x^2-8xy+16y^2 & x^2-6xy+8y^2 \\
\underline{x^2-6xy+\ 8y^2} & \underline{1\qquad\qquad\qquad} \\
-2y)\underline{-2xy+\ 8y^2} & \\
x^2-6xy+8y^2 & \underline{x-\ 4y,\ \text{H. C. D.}} \\
\underline{x^2-6xy+8y^2} & \underline{x-\ 2\qquad\qquad}
\end{array}$$

10. $(x^3-y^3)\div(x^2-2xy+y^2)=x-y.$

$(x^2-2xy+y^2)\div(x-y)=x-y.$

Therefore, $x-y$ is the H. C. D.

11. $x^4-2x^2+1=(x^2-1)(x^2-1)=(x+1)(x-1)(x+1)(x-1);$

$$x^4-4x^3+6x^2-4x+1=(x-1)^4$$
$$=(x-1)(x-1)(x-1)(x-1).$$

Therefore, $(x-1)(x-1)$, or, x^2-2x+1 is the H.C.D.

12.

$$\begin{array}{r|l} 6x^3+6x^2-6x-6 & \underline{2x^3+6x^2+6x+2} \\ \underline{6x^3+18x^2+18x+6} & \underline{3\qquad\qquad\quad} \end{array}$$
$$-6)\underline{-12x^2-24x-12}$$
$$\begin{array}{r|l} 2x^3+6x^2+6x+2 & \underline{2x^2+4x+2,\ \text{H.C.D.}} \\ \underline{2x^3+6x^2+6x+2} & \underline{x+1.\qquad\qquad} \end{array}$$

13.

$$\begin{array}{r|l} 3x^4+3x^3-21x^2-9x & \underline{3x^3+3x^2-15x+9} \\ \underline{3x^4+3x^3-15x^2-9x} & \underline{x\qquad\qquad\qquad} \end{array}$$
$$-2x)\underline{-6x^2-18x}$$
$$\begin{array}{r|l} 3x^3+3x^2-15x+9 & \underline{3x+9,\ \text{H.C.D.}} \\ \underline{3x^3+3x^2-15x+9} & \underline{x^2-2x+1.} \end{array}$$

14.

$$\begin{array}{r|l} 25x^4+5x^3-x-1 & \underline{20x^4+x^2-1} \\ \underline{4\qquad\qquad\qquad} & \underline{5\qquad\qquad} \end{array}$$
$$100x^4+20x^3-4x-4$$
$$\underline{100x^4+5x^2\qquad-5}$$
$$\begin{array}{r|l} 20x^4+x^2-1 & \underline{20x^3-5x^2-4x+1} \\ \underline{20x^4-5x^3-4x^2+x} & \underline{x+1\qquad\qquad} \end{array}$$
$$5x^3+5x^2-x-1$$
$$\underline{4\qquad\qquad\qquad}$$
$$20x^3+20x^2-4x-4$$
$$\underline{20x^3-5x^2-4x+1}$$
$$5)\underline{25x^2-5}$$
$$\begin{array}{r|l} 20x^3-5x^2-4x+1 & \underline{5x^2-1,\ \text{H.C.D.}} \\ \underline{20x^3-5x^2-4x+1} & \underline{4x-1\qquad\qquad} \end{array}$$

15.

$$x^2-9=(x+3)(x-3);$$
$$x^2-3x-18=(x+3)(x-6);$$
$$x^2+11x+24=(x+3)(x+8).$$

Therefore, $x+3$ is the H.C.D.

16.

$$x^2-3x-28=(x-7)(x+4);$$
$$x^2-11x+28=(x-7)(x-4);$$
$$x^2-15x+56=(x-7)(x-8).$$

Therefore, $x-7$ is the H.C.D.

17.

$$x^2+6x+9=(x+3)(x+3);$$

$$x^3-x^2-12x=x(x+3)(x-4);$$

$$x^2-4x-21=(x+3)(x-7).$$

Therefore, $x+3$ is the H. C. D.

18.

$$a^4-b^4=(a^2+b^2)(a^2-b^2);$$

$$a^3+a^2b-ab^2-b^3=a^2(a+b)-b^2(a+b)=(a^2-b^2)(a+b);$$

$$a^4-2a^2b^2+b^4=(a^2-b^2)(a^2-b^2).$$

Therefore, a^2-b^2 is the H. C. D.

19.

$$x^2)\underline{x^4+5x^3+6x^2}$$

$$\begin{array}{r|l}
x^3+3x^2+3x+2 & \underline{x^2+5x+6} \\
\underline{x^3+5x^2+6x} & \underline{x-2} \\
-2x^2-3x+2 & \\
\underline{-2x^2-10x-12} & \\
7)\underline{7x+14} &
\end{array}$$

$$\begin{array}{r|l}
x^2+5x+6 & \underline{x+2}, \text{ H. C. D. of first two.} \\
\underline{x^2+5x+6} & \underline{x+3}
\end{array}$$

$$\begin{array}{r|l}
3x^3+8x^2+5x+2 & \underline{x+2}, \text{ H. C. D. of all.} \\
\underline{3x^3+6x^2} & \underline{3x^2+2x+1} \\
2x^2+5x+2 & \\
\underline{2x^2+4x} & \\
x+2. &
\end{array}$$

20. $a^3+3a^2b+3ab^2+b^3=(a+b)^3=(a+b)(a+b)(a+b);$

$$a^2-b^2=(a+b)(a-b).$$

Therefore, $(a+b)$ is the H. C. D. of first and third quantities.

$$\begin{array}{r|l}
4b^2)\underline{4a^2b^2+12ab^3+8b^4} & \underline{a+b}, \text{ H. C. D. of all.} \\
a^2+3ab+2b^2 & \underline{a+2b} \\
\underline{a^2+3ab+2b^2} &
\end{array}$$

21.

$$\begin{array}{r|l}
9x^4+12x^3+10x^2+4x+1 & \underline{3x^4+8x^3+14x^2+8x+3} \\
\underline{9x^4+24x^3+42x^2+24x+9} & \underline{3} \\
-4)\underline{-12x^3-32x^2-20x-8} &
\end{array}$$

$$\begin{array}{r|l}
3x^4+8x^3+14x^2+8x+3 & \underline{3x^3+8x^2+5x+2} \\
\underline{3x^4+8x^3+5x^2+2x} & \underline{x} \\
3)\underline{9x^2+6x+3} &
\end{array}$$

$$\begin{array}{r|l}
3x^3+8x^2+5x+2 & \underline{3x^2+2x+1}, \text{ H. C. D.} \\
\underline{3x^3+8x^2+5x+2} & \underline{x+2}
\end{array}$$

Page 95.

22.

$$\begin{array}{r|l}
x^5+0x^4+6x^3+6x^2+8x+24 & x^4+3x^3+9x^2+12x+20 \\ \cline{2-2}
\underline{x^5+3x^4+9x^3+12x^2+20x} & x-3 \\ \hline
-3x^4-3x^3-6x^2-12x+24 & \\
\underline{-3x^4-9x^3-27x^2-36x-60} & \\
3)\underline{6x^3+21x^2+24x+84} &
\end{array}$$

$$\begin{array}{l|l}
x^4+3x^3+9x^2+12x+20 & 2x^3+7x^2+8x+28 \\ \cline{2-2}
2 & x-1 \\ \hline
2x^4+6x^3+18x^2+24x+40 & \\
\underline{2x^4+7x^3+8x^2+28x} & \\
-x^3+10x^2-4x+40 & \\
\underline{2} & \\
-2x^3+20x^2-8x+80 & \\
\underline{-2x^3-7x^2-8x-28} & \\
27)\underline{27x^2+108} &
\end{array}$$

$$\begin{array}{l|l}
2x^3+7x^2+8x+28 & x^2+4, \text{ H. C. D.} \\ \cline{2-2}
\underline{2x^3+7x^2+8x+28} & 2x+7.
\end{array}$$

23.

$$\begin{array}{l|l}
3a^2x^2+a^2x+2a^2+12x^2+4x+8 & a^2x^2+3a^2x+4a^2+4x^2+12x+16 \\ \cline{2-2}
\underline{3a^2x^2+9a^2x+a^2+12x^2+36x+4218} & 3 \\ \hline
-2)\underline{-8a^2x-10a^2-32x-40} &
\end{array}$$

$$\begin{array}{l|l}
a^2x^2+3a^2x+4a^2+4x^2+12x+16 & 4a^2x+5a^2+16x+20 \\ \cline{2-2}
4 & x+7 \\ \hline
4a^2x^2+12a^2x+16a^2+16x^2+48x+64 & \\
\underline{4a^2x^2+5a^2x+16x^2+20x} & \\
7a^2x+16a^2+28x+64 & \\
\underline{4} & \\
28a^2x+64a^2+112x+256 & \\
\underline{28a^2x+35a^2+112x+140} & \\
29)\underline{29a^2+116} &
\end{array}$$

$$\begin{array}{l|l}
4a^2x+5a^2+16x+20 & a^2+4, \text{ H. C. D.} \\ \cline{2-2}
\underline{4a^2x+5a^2+16x+20} & 4x+5.
\end{array}$$

24.

$$\begin{array}{r|l}
 & 3)15x^3+9x^2-3x+3 \\
40x^3-21x^2+10x-1 & 5x^3+3x^2-x+1 \\
40x^3+24x^2-8x+8 & 8 \\ \hline
-9)-45x^2+18x-9 &
\end{array}$$

$$\begin{array}{r|l}
5x^3+3x^2-x+1 & 5x^2-2x+1,\ \text{H. C. D.} \\
5x^3-2x^2+x & x+1 \\ \hline
5x^2-2x+1 & \\
5x^2-2x+1 & \\ \hline
\end{array}$$

25. Reserving the common factor a.

$$\begin{array}{r|l}
 & 3)\ 3x^2-15x+12 \\
16x^3-20x^2+10x-6 & x^2-5x+4 \\
16x^3-80x^2+64x & 16x+10 \\ \hline
6)60x^2-54x-6 & \\
10x^2-9x-1 & \\
10x^2-50x+40 & \\ \hline
41)41x-41 &
\end{array}$$

$$\begin{array}{r|l}
x^2-5x+4 & x-1 \\
x^2-x & x+4 \\ \hline
-4x+4 & \\
-4x+4 & \\ \hline
\end{array}$$

$\therefore\ a(x-1)$ is H. C. D.

26. Reserving the common factor 2.

$$\begin{array}{r|l}
2a^3+2a^2b-2ab^2-2b^3 & a^3-a^2b-ab^2+b^3 \\
2a^3-2a^2b-2ab^2+2b^3 & 2 \\ \hline
4b)4a^2b-4b^3 &
\end{array}$$

$$\begin{array}{r|l}
a^3-a^2b-ab^2+b^3 & a^2-b^2 \\
a^3-ab^2 & a-b \\ \hline
-a^2b+b^3 & \\
-a^2b+b^3 & \\ \hline
\end{array}$$

$\therefore\ 2(a^2-b^2)$ is H. C. D.

27.

$$\begin{array}{r|l}
4x^3-25x^2+20x+25 & 2x^2-14x+20 \\
4x^3-28x^2+40x & 2x+3 \\ \hline
3x^2-20x+25 & \\
2 & \\ \hline
6x^2-40x+50 & \\
6x^2-42x+60 & \\ \hline
2)2x-10 &
\end{array}$$

$$\begin{array}{r|l}
2x^2-14x+20 & x-5 = \text{H. C. D.} \\
2x^2-10x & 2x-4 \\ \hline
-4x+20 & \\
-4x+20 & \\ \hline
\end{array}$$

28. Reserving the common factor 3.

$$\begin{array}{r|l}
2x^3+18x^2-46x-22 & x^2+7x-44 \\
2x^3+14x^2-88x & 2x+4 \\ \hline
4x^2+42x-22 & \\
4x^2+28x-176 & \\ \hline
14)\ 14x+154 & \\
\end{array}$$

$$\begin{array}{r|l}
x^2+7x-44 & x+11 \\
x^2+11x & x-4 \\ \hline
-4x-44 & \\
-4x-44 & \\ \hline
\end{array}$$

$\therefore\ 3(x+11)$ is H. C. D.

29.

$$\begin{array}{r|l}
x^5-x^4-3x^3+x^2+3x+1 & x^4-x^3-3x^2+2x+2 \\
x^5-x^4-3x^3+2x^2+2x & x \\ \hline
-1)\ -x^2+x+1 & \\
\end{array}$$

$$\begin{array}{r|l}
x^4-x^3-3x^2+2x+2 & x^2-x-1 = \text{H. C. D.} \\
x^4-x^3-x^2 & x^2-2 \\ \hline
-2x^2+2x+2 & \\
-2x^2+2x+2 & \\ \hline
\end{array}$$

30.

$$\begin{array}{r|l}
a^3-6a^2-86a+35 & a^3-5a^2-99a+40 \\
a^3-5a^2-99a+40 & 1 \\ \hline
-1)\ -a^2+13a-5 & \\
\end{array}$$

$$\begin{array}{r|l}
a^3-5a^2-99a+40 & a^2-13a+5 = \text{H. C. D.} \\
a^3-13a^2+5a & a+8 \\ \hline
8a^2-104a+40 & \\
8a^2-104a+40 & \\ \hline
\end{array}$$

31.

$$\begin{array}{r|l}
14x^4+x^3+8x^2-x+2 & \\
3 & \\ \hline
42x^4+3x^3+24x^2-3x+6 & 6x^4+7x^3+7x^2+3x+1 \\
42x^4+49x^3+49x^2+21x+7 & 7 \\ \hline
-1)\ -46x^3+25x^2-24x-1 & \\
\end{array}$$

$$\begin{array}{r|l}
6x^4+7x^3+7x^2+3x+1 & \\
23 & 46x^3+25x^2+24x+1 \\ \hline
138x^4+161x^3+161x^2+69x+23 & 3x+43 \\
138x^4+75x^3+72x^2+3x & \\ \hline
86x^3+89x^2+66x+23 & \\
23 & \\ \hline
1978x^3+2047x^2+1518x+529 & \\
1978x^3+1075x^2+1032x+43 & \\ \hline
486)\ 972x^2+486x+486 & \\
\end{array}$$

$$\begin{array}{r|l}
46x^3+25x^2+24x+1 & 2x^2+x+1 = \text{H. C. D.} \\
46x^3+23x^2+23x & 23x+1 \\ \hline
2x^2+x+1 & \\
2x^2+x+1 & \\ \hline
\end{array}$$

32.

$$\begin{array}{r|l}
3x^3 - x^2 - 2x - 16 & \\
2 & \\
\hline
6x^3 - 2x^2 - 4x - 32 & 2x^3 - 2x^2 - 3x - 2 \\
\cline{2-2}
6x^3 - 6x^2 - 9x - 6 & 3 \\
\hline
\multicolumn{1}{|r}{4x^2 + 5x - 26} & \\
\cline{1-1}
\multicolumn{1}{|l}{x + 9} & \\
\cline{1-1}
\end{array}$$

$$\begin{array}{r|l}
2x^3 - 2x^2 - 3x - 2 & \\
2 & \\
\hline
4x^3 - 4x^2 - 6x - 4 & \\
4x^3 + 5x^2 - 26x & \\
\hline
-9x^2 + 20x - 4 & \\
-4 & \\
\hline
36x^2 - 80x + 16 & \\
36x^2 + 45x - 234 & \\
\hline
-125)\underline{-125x + 250} & \\
4x^2 + 5x - 26 & x - 2 = \text{H. C. D.} \\
\cline{2-2}
4x^2 - 8x & 4x + 13 \\
\hline
13x - 26 & \\
\underline{13x - 26} &
\end{array}$$

COMMON MULTIPLES.

Page 97.

8.
$$x^2 - y^2 = (x + y)(x - y);$$
$$x^2 - 2xy + y^2 = (x - y)(x - y).$$

Therefore, $(x - y)(x - y)(x + y)$, or $x^3 - x^2y - xy^2 + y^3 =$ L. C. M.

9.
$$x^2 - y^2 = (x + y)(x - y);$$
$$x^2 + 2xy + y^2 = (x + y)(x + y).$$

Therefore, $(x + y)(x + y)(x - y)$, or $x^3 + x^2y - xy^2 - y^3 =$ L. C. M.

10.
$$x^2 - y^2 = (x + y)(x - y);$$
$$x^2 - 2xy + y^2 = (x - y)(x - y);$$
$$x^2 + 2xy + y^2 = (x + y)(x + y).$$

Therefore, $(x+y)(x+y)(x-y)(x-y)$, or $x^4 - 2x^2y^2 + y^4 =$ L. C. M.

11.
$$x^2 - y^2 = (x + y)(x - y);$$
$$x^3 - y^3 = (x - y)(x^2 + xy + y^2).$$

Therefore, $(x-y)(x^2+xy+y^2)(x+y)$, or $x^4 + x^3y - xy^3 - y^4 =$ L. C. M.

12.
$$a^2(x - z);$$
$$y^2(x^2 - z^2) = y^2(x + z)(x - z).$$

Therefore, $a^2(x - z) \times y^2(x + z)$, or $a^2y^2(x^2 - z^2) =$ L. C. M.

13.
$$x^2 - 1 = x^2 - 1;$$
$$x^2 + 1 = x^2 + 1;$$
$$x^4 - 1 = (x^2 + 1)(x^2 - 1).$$

Therefore, $(x^2 + 1)(x^2 - 1)$, or $x^4 - 1 =$ L. C. M.

14.

$$2x(x-y)=2x(x-y);$$
$$4xy(x^2-y^2)=2y\times 2x(x+y)(x-y);$$
$$6xy^2(x+y)=3y^2\times 2x(x+y).$$

Therefore, $12xy^2(x^2-y^2)=$ L. C. M.

15.

$$x^2-x=x(x-1);$$
$$x^3-1=(x-1)(x^2+x+1);$$
$$(x^3+1)=(x+1)(x^2-x+1).$$

Therefore, $x(x-1)(x+1)(x^2+x+1)(x^2-x+1)$, or $x(x^6-1)=$ L. C. M.

16.

$$x^2-x=x(x-1);$$
$$x^2-1=(x-1)(x+1);$$
$$x^3-1=(x-1)(x^2+x+1).$$

Therefore, $x(x-1)(x+1)(x^2+x+1)$, or $x(x^4+x^3-x-1)=$ L.C.M.

17. The L. C. M. of the coefficients is 8, and of the other quantities is $1-x^2$.

Therefore, $8(1-x^2)=$ L. C. M.

18.

$$x^2+5x+6=(x+3)(x+2);$$
$$x^2+6x+8=(x+4)(x+2).$$

Therefore, $(x+2)(x+3)(x+4)$, or $x^3+9x^2+26x+24=$ L. C. M.

19.

$$a^2-a-20=(a-5)(a+4);$$
$$a^2+a-12=(a-3)(a+4).$$

Therefore, $(a+4)(a-5)(a-3)$, or $a^3-4a^2-17a+60=$ L. C. M.

20.

$$x^2-9x-22=(x-11)(x+2);$$
$$x^2-13x+22=(x-11)(x-2).$$

Therefore, $(x+2)(x-2)(x-11)$, or $x^3-11x^2-4x+44=$ L. C. M.

21.

$$x^2-8x+15=(x-3)(x-5);$$

x^2+2x-5 cannot be factored.

Therefore, $(x-3)(x-5)(x^2+2x-5)$, or $x^4-6x^3-6x^2+70x-75=$ L. C. M.

22.

$$x^3+x^2y+xy^2+y^3=(x^2+y^2)(x+y);$$
$$x^3-x^2y+xy^2-y^3=(x^2+y^2)(x-y).$$

Therefore, $(x^2+y^2)(x+y)(x-y)$, or $x^4-y^4=$ L.C.M.

Page 98.

23.
$$\begin{aligned} x^3-x^2y+xy^2-y^3 &= (x^2+y^2)(x-y); \\ x^3+x^2y-xy^2-y^3 &= (x-y)(x+y)(x+y). \end{aligned}$$

Therefore, $(x^2+y^2)(x-y)(x+y)(x+y)$, or $x^5-xy^4+x^4y-y^5=$ L. C. M.

24.
$$\begin{aligned} a^3-2a^2+4a-8 &= (a^2+4)(a-2); \\ a^3+2a^2-4a-8 &= (a^2-4)(a+2) \\ &= (a+2)(a-2)(a+2). \end{aligned}$$

Therefore, $(a^2+4)(a+2)(a-2)(a+2)$, or $a^5+2a^4-16a-32=$ L. C. M.

25.
$$\begin{aligned} x^2+y^2 &= x^2+y^2; \\ x^3-xy^2 &= x(x+y)(x-y); \\ x^3+xy^2+x^2y+y^3 &= (x^2+y^2)(x+y). \end{aligned}$$

Therefore, $x(x+y)(x-y)(x^2+y^2)$, or $x^5-xy^4=$ L. C. M.

26.
$$\begin{aligned} x^2-4 &= (x+2)(x-2); \\ x^2-x-6 &= (x+2)(x-3); \\ x^3-3x^2-4x+12 &= (x+2)(x-2)(x-3). \end{aligned}$$

Therefore, $(x+2)(x-2)(x-3)$, or $x^3-3x^2-4x+12=$ L. C. M.

27.
$$\begin{aligned} x-5 &= x-5; \\ x^2-2ax+a^2 &= (x-a)(x-a); \\ x^2-10x+25 &= (x-5)(x-5); \\ x^2+5a-5x-ax &= (x-a)(x-5). \end{aligned}$$

Therefore, $(x-a)(x-a)(x-5)(x-5)=$ L. C. M.

28.
$$\begin{aligned} x^4-16 &= (x^2+4)(x+2)(x-2); \\ x^2+4x+4 &= (x+2)(x+2); \\ (x^2-4) &= (x+2)(x-2). \end{aligned}$$

Therefore, $(x^2+4)(x+2)(x-2)(x+2)$, or $x^5+2x^4-16x-32=$ L. C. M.

29.
$$\begin{aligned} m^2-3m+2 &= (m-2)(m-1); \\ m^2-4m+4) &= (m-2)(m-2). \end{aligned}$$

Therefore, $(m-2)(m-2)(m-1)$,

or $m^3-5m^2+8m-4=$ L. C. M.

30.
$$\begin{aligned} p^2-7p+10 &= (p-5)(p-2); \\ p^2-p-20 &= (p-5)(p+4). \end{aligned}$$

Therefore, $(p-5)(p-4)(p-2)$,

or $p^3-3p^2-18p+40=$ L. C. M.

31. $5c - c^2 - 4 = (c - 4)(1 - c);$
$c^2 - 4c + 3 = (c - 3)(c - 1).$
Therefore, $(c - 4)(1 - c)(c - 3),$
or $8c^2 - c^3 - 19c + 12 = \text{L. C. M.}$

32. $1 - p + p^2$ cannot be factored;
$1 + p - p^2$ cannot be factored;
$1 - p^2 = (1 - p)(1 + p).$
Therefore, $(1 - p + p^2)(1 + p - p^2)(1 - p^2),$
or $1 - 2p^2 + 2p^3 - 2p^5 + p^6 = \text{L. C. M.}$

33. $4ac - 4c - 4c^2 = 4c(a - c - 1);$
$2ab - 2bc - 2b = 2b(a - c - 1).$
Therefore, $4bc(a - c - 1) = \text{L. C. M.}$

34. $3x^2 - 19x + 6 = (x - 6)(3x - 1);$
$x^2 - 10x + 24 = (x - 6)(x - 4).$
Therefore, $(x - 6)(x - 4)(3x - 1),$
or $3x^3 - 31x^2 + 82x - 24 = \text{L. C. M.}$

35. $m^2 - m - 90 = (m - 10)(m + 9);$
$m^2 - 13m + 30 = (m - 10)(m - 3);$
$m^2 - 100 = (m - 10)(m + 10).$
Therefore, $(m - 10)(m + 10)(m + 9)(m - 3),$
or $m^4 + 6m^3 - 127m^2 - 600m + 2700 = \text{L. C. M.}$

36. $y^2 - 2y - 35 = (y - 7)(y + 5);$
$y^2 - 8y + 7 = (y - 7)(y - 1);$
$y^2 + 4y - 5 = (y + 5)(y - 1).$
Therefore, $(y - 7)(y + 5)(y - 1),$
or $y^3 - 3y^2 - 33y + 35 = \text{L. C. M.}$

37. $2x^3 - 12x^2 + 19x - 12 = (x - 4)(2x^2 - 4x + 3);$
$2x^3 - 6x^2 + 7x - 3 = (x - 1)(2x^2 - 4x + 3).$
Therefore, $(x - 4)(x - 1)(2x^2 - 4x + 3),$
or $2x^4 - 14x^3 + 31x^2 - 31x + 12 = \text{L. C. M.}$

38. $a^4 - 1 = (a^2 - 1)(a^2 + 1);$
$a^4 + 2a^3 - 2a^2 - 2a + 1 = (a^2 - 1)(a^2 + 2a - 1).$
Therefore, $(a^4 - 1)(a^2 + 2a - 1),$
or $a^6 + 2a^5 - a^4 - a^2 - 2a + 1 = \text{L. C. M.}$

39. $c^2 - 5c + 4 = (c - 4)(c - 1);$
$c^2 - 6c + 8 = (c - 4)(c - 2);$
$c^2 - 8c + 16 = (c - 4)(c - 4).$
Therefore, $(c - 4)(c - 4)(c - 2)(c - 1) = \text{L.C.M.}$

40. $a^2 + 3a - 4 = (a + 4)(a - 1);$
$a^2 - 6a + 5 = (a - 5)(a - 1);$
$a^2 - a - 20 = (a - 5)(a + 4).$
Therefore, $(a + 4)(a - 5)(a - 1) = \text{L.C.M.}$

41. $x^2 - (y + z)^2 = (x - y - z)(x + y + z);$
$y^2 - (x + z)^2 = (y - x - z)(x + y + z);$
$z^2 - (x + y)^2 = (z - x - y)(x + y + z).$
Therefore, $(x + y + z)(x - y - z)(y - x - z)(z - x - y) = \text{L.C.M.}$

42. $a^3 - 8a^2 + 19a - 12 = (a - 4)(a - 3)(a - 1);$
$a^3 - 9a^2 + 26a - 24 = (a - 2)(a - 3)(a - 4);$
$a^3 - 6a^2 + 11a - 6 = (a - 2)(a - 3)(a - 1).$
Therefore, $(a - 1)(a - 2)(a - 3)(a - 4) = \text{L.C.M.}$

43. $4c^2 - 9 = (2c - 3)(2c + 3);$
$6c^2 - 5c - 6 = (2c - 3)(3c + 2);$
$6c^2 + 13c + 6 = (3c + 2)(2c + 3).$
Therefore, $(2c - 3)(2c + 3)(3c + 2) = \text{L.C.M.}$

44. $1 - x + x^2$ cannot be factored;
$1 + x + x^2$ cannot be factored;
$1 + x^2 + x^4 = (1 - x + x^2)(1 + x + x^2).$
Therefore, $1 + x^2 + x^4 = \text{L.C.M.}$

45. $x^3 + 3x^2y + 3xy^2 + y^3 = (x + y)(x + y)(x + y);$
$x^3 - xy^2 + x^2y - y^3 = (x + y)(x + y)(x - y).$
Therefore. $(x + y)^3(x - y),$
or $x^4 + 2x^3y - 2xy^3 - y^4 = \text{L.C.M.}$

46. $(a + b)(a + c)(b + c) = \text{L.C.M.}$

47. $a^2 + ab + b^2$ cannot be factored;
$a^3 - b^3 = (a - b)(a^2 + ab + b^2);$
$a - b = a - b.$
Therefore, $a^3 - b^3 = \text{L.C.M.}$

48.
$$4a^3(a+x) = 2 \times 2 \times a^2 \times a(a+x);$$
$$4a^3(a-x) = 2 \times 2 \times a^2 \times a(a-x);$$
$$2a^2(a^2+x^2) = 2 \times a^2(a^2+x^2).$$

Therefore, $4a^3(a^2+x^2)(a^2-x^2) = \text{L.C.M.}$

REDUCTION OF FRACTIONS.

Page 103.

20. $$\frac{a^2-b^2}{a^2-2ab+b^2} = \frac{(a+b)(a-b)}{(a-b)(a-b)} = \frac{a+b}{a-b}.$$

21. $$\frac{a^2-b^2}{a^2+2ab+b^2} = \frac{(a+b)(a-b)}{(a+b)(a+b)} = \frac{a-b}{a+b}.$$

23. $$\frac{2a^2-2}{a^2-2a+1} = \frac{2(a-1)(a+1)}{(a-1)(a-1)} = \frac{2(a+1)}{a-1}.$$

24. $$\frac{3a^2-4ab+b^2}{a^2-ab} = \frac{(3a-b)(a-b)}{a(a-b)} = \frac{3a-b}{a}.$$

25. $$\frac{x^2-z^2}{x^3-z^3} = \frac{(x-z)(x+z)}{(x-z)(x^2+xz+z^2)} = \frac{x+z}{x^2+xz+z^2}.$$

26. $$\frac{a^2-2ab}{a^2-4ab+4b^2} = \frac{a(a-2b)}{(a-2b)(a-2b)} = \frac{a}{a-2b}.$$

27. $$\frac{2x^4y^4+2}{3x^8y^8-3} = \frac{2(x^4y^4+1)}{3(x^8y^8-1)} = \frac{2}{3(x^4y^4-1)}.$$

28. $$\frac{18a^2c-6bc}{42a^2d-14bd} = \frac{3c(6a^2-2b)}{7d(6a^2-2b)} = \frac{3c}{7d}.$$

29. $$\frac{(x+y)^2}{x^2-y^2} = \frac{(x+y)(x+y)}{(x+y)(x-y)} = \frac{x+y}{x-y}.$$

30. $$\frac{m^3+n^3}{(m+n)^3} = \frac{(m+n)(m^2-mn+n^2)}{(m+n)(m+n)(m+n)} = \frac{m^2-mn+n^2}{m^2+2mn+n^2}.$$

31. $$\frac{(1-x^2)^2}{(1+x)^3} = \frac{(1+x)(1-x)(1+x)(1-x)}{(1+x)(1+x)(1+x)} = \frac{(1-x)^2}{1+x}.$$

32. $$\frac{a^2-4a-12}{a^2+5a+6} = \frac{(a-6)(a+2)}{(a+3)(a+2)} = \frac{a-6}{a+3}.$$

33. $$\frac{x^2-5x+6}{x^2+4x-21} = \frac{(x-3)(x-2)}{(x+7)(x-3)} = \frac{x-2}{x+7}.$$

34. $\frac{a^2 + a - 90}{a^2 + 8a - 20} = \frac{(a + 10)(a - 9)}{(a + 10)(a - 2)} = \frac{a - 9}{a - 2}.$

35. $\frac{2x^2 - 5x + 3}{x^2 + x - 2} = \frac{(2x - 3)(x - 1)}{(x + 2)(x - 1)} = \frac{2x - 3}{x + 2}.$

36. $\frac{m^2 - 12m + 32}{m^2 + 2m - 24} = \frac{(m - 8)(m - 4)}{(m + 6)(m - 4)} = \frac{m - 8}{m + 6}$

37. $\frac{3a^2 - 6a}{2ax - 4x} = \frac{3a(a - 2)}{2x(a - 2)} = \frac{3a}{2x}.$

38. $\frac{7a^2b + 7abx}{a^2 - x^2} = \frac{7ab(a + x)}{(a - x)(a + x)} = \frac{7ab}{a - x}.$

39. $\frac{x^2 - 1}{x^2 + 2x + 1} = \frac{(x + 1)(x - 1)}{(x + 1)(x + 1)} = \frac{x - 1}{x + 1}.$

40. $\frac{x^2 - 1}{2xy + 2y} = \frac{(x + 1)(x - 1)}{2y(x + 1)} = \frac{x - 1}{2y}.$

41. $\frac{x^3 - a^2x}{x^2 - 2ax + a^2} = \frac{x(x - a)(x + a)}{(x - a)(x - a)} = \frac{x(x + a)}{x - a}.$

Page 104.

42. $\frac{x^5 - x^3y^2}{x^4 - y^4} = \frac{x^3(x^2 - y^2)}{(x^2 + y^2)(x^2 - y^2)} = \frac{x^3}{x^2 + y^2}.$

43. $\frac{x^2 + 6x + 9}{x^3 - x^2 - 12x} = \frac{(x + 3)(x + 3)}{x(x - 4)(x + 3)} = \frac{x + 3}{x^2 - 4x}.$

44. $\frac{x^2 - 3x - 28}{x^2 - 11x + 28} = \frac{(x - 7)(x + 4)}{(x - 7)(x - 4)} = \frac{x + 4}{x - 4}.$

45. $\frac{m^2 - n^2}{m^3 - n^3} = \frac{(m - n)(m + n)}{(m - n)(m^2 + mn + n^2)} = \frac{m + n}{m^2 + mn + n^2}.$

46. $\frac{a^6 + b^6}{a^4 - b^4} = \frac{(a^4 - a^2b^2 + b^4)(a^2 + b^2)}{(a^2 - b^2)(a^2 + b^2)} = \frac{a^4 - a^2b^2 + b^4}{a^2 - b^2}.$

47. $\frac{m^2 - n^2}{m^3 + n^3} = \frac{(m + n)(m - n)}{(m + n)(m^2 - mn + n^2)} = \frac{m - n}{m^2 - mn + n^2}.$

48. $\frac{m^8 - n^8}{m^6 - n^6} = \frac{(m^4 + n^4)(m^2 + n^2)(m^2 - n^2)}{(m^4 + m^2n^2 + n^4)(m^2 - n^2)} = \frac{(m^4 + n^4)(m^2 + n^2)}{m^4 + m^2n^2 + n^4}.$

49. $\frac{9a^2b + 9ab^2}{3a^2 + 6ab + 3b^2} = \frac{3ab\,(3a + 3b)}{(a + b)(3a + 3b)} = \frac{3ab}{a + b}.$

50. $\dfrac{x^3 - y^3}{x^2 - 2xy + y^2} = \dfrac{(x - y)(x^2 + xy + y^2)}{(x - y)(x - y)} = \dfrac{x^2 + xy + y^2}{x - y}.$

51. $\dfrac{y^3 - b^2y}{y^2 + 2by + b^2} = \dfrac{y(y + b)(y - b)}{(y + b)(y + b)} = \dfrac{y(y - b)}{y + b}.$

52. $\dfrac{a^2 - 4b^2}{(a - 2b)^2} = \dfrac{(a - 2b)(a + 2b)}{(a - 2b)(a - 2b)} = \dfrac{a + 2b}{a - 2b}.$

53. $\dfrac{a^2 + 2a - 3}{a^2 + 5a + 6} = \dfrac{(a + 3)(a - 1)}{(a + 3)(a + 2)} = \dfrac{a - 1}{a + 2}.$

54. $\dfrac{a^2 + 5a + 6}{a^2 + 7a + 12} = \dfrac{(a + 3)(a + 2)}{(a + 3)(a + 4)} = \dfrac{a + 2}{a + 4}.$

55. $\dfrac{6c^2 - 5c - 6}{8c^2 - 2c - 15} = \dfrac{(3c + 2)(2c - 3)}{(4c + 5)(2c - 3)} = \dfrac{3c + 2}{4c + 5}.$

56. $\dfrac{a^2 - (b + c)^2}{a^2 + ab + ac} = \dfrac{(a - b - c)(a + b + c)}{a(a + b + c)} = \dfrac{a - b - c}{a}.$

57. $\dfrac{15x^2 + xy - 2y^2}{9x^2 + 3xy - 2y^2} = \dfrac{(5x + 2y)(3x - y)}{(3x + 2y)(3x - y)} = \dfrac{5x + 2y}{3x + 2y}.$

58. $\dfrac{a^2 - 12a + 35}{a^2 - 15a + 56} = \dfrac{(a - 7)(a - 5)}{(a - 7)(a - 8)} = \dfrac{a - 5}{a - 8}.$

59. $\dfrac{x^3 + 5x^2 + 5x + 1}{x^3 + 1} = \dfrac{(x + 1)(x^2 + 4x + 1)}{(x + 1)(x^2 - x + 1)} = \dfrac{x^2 + 4x + 1}{x^2 - x + 1}.$

Page 108.

18. The lowest common denominator is $x^4 - 1$. The fractions become

$$\frac{x^4 - 2x^2 + 1}{x^4 - 1},\quad \frac{x^4 + 2x^2 + 1}{x^4 - 1},\quad \frac{x^4 + 1}{x^4 - 1}.$$

19. The lowest common denominator is $(a - b)(b - c)(a - c)$. The fractions become

$$\frac{a - c}{(a - b)(b - c)(a - c)},\quad \frac{b - c}{(a - b)(b - c)(a - c)}.$$

Page 109.

20. The lowest common denominator is

$$(a^2 - 9)(a + 1),\ \text{or}\ a^3 + a^2 - 9a - 9.$$

The fractions become

$$\frac{a^3 - 9a}{a^3 + a^2 - 9a - 9},\quad \frac{a^2 - 2a - 3}{a^3 + a^2 - 9a - 9},\quad \frac{2a + 2}{a^3 + a^2 - 9a - 9},\quad \frac{a^3 + 3a^2 - a - 3}{a^3 + a^2 - 9a - 9}.$$

21. The lowest common denominator is

$$6x + 12.$$

The fractions become

$$\frac{18x - 6}{6x + 12}, \quad \frac{6x + 15}{6x + 12}, \quad \frac{4x - 1}{6x + 12}.$$

22. The lowest common denominator is

$$(4x^2 - 25)(x - 5), \text{ or } 4x^3 - 20x^2 - 25x + 125.$$

The fractions become

$$\frac{32x^2 - 200}{4x^3 - 20x^2 - 25x + 125}, \quad \frac{25x - 125}{4x^3 - 20x^2 - 25x + 125}, \quad \frac{6x^3 - 45x^2 + 75x}{4x^3 - 20x^2 - 25x + 125}.$$

23. The lowest common denominator is

$$(m^3 - n^3)(m + n), \text{ or } m^4 - mn^3 + m^3n - n^4.$$

The fractions become

$$\frac{m^3 + 2m^2n + 2mn^2 + n^3}{m^4 - mn^3 + m^3n - n^4}, \quad \frac{2m^3 + 2m^2n + 2mn^2}{m^4 - mn^3 + m^3n - n^4}, \quad \frac{m^2 + 2mn + n^2}{m^4 - mn^3 + m^3n - n^4}.$$

24. The lowest common denominator is

$(x - 1)(x + 2)(x - 3)(x^2 + 2x + 4)$, or $x^5 - 5x^3 - 12x^2 - 8x + 24$.

The fractions become

$$\frac{x^4 + 2x^3 - 5x^2 - 18x - 36}{x^5 - 5x^3 - 12x^2 - 8x + 24}, \quad \frac{x^4 + 2x^3 + 3x^2 - 2x - 4}{x^5 - 5x^3 - 12x^2 - 8x + 24},$$

$$\frac{x^4 + 2x^3 - 8x - 16}{x^5 - 5x^3 - 12x^2 - 8x + 24}, \quad \frac{x^3 - 2x^2 - 5x + 6}{x^5 - 5x^3 - 12x^2 - 8x + 24}.$$

FRACTIONAL EQUATIONS.

Page 112.

33. $$\frac{x + 1}{3} - \frac{3x - 1}{5} = x - 2. \qquad (1)$$

Clearing, $5x + 5 - 9x + 3 = 15x - 30.$ (2)

Transposing, etc., $-19x = -38;$ (3)

$$x = 2.$$

34. $$\frac{x - 3}{4} - \frac{x - 1}{9} = \frac{x - 5}{6}. \qquad (1)$$

Clearing, $9x - 27 - 4x + 4 = 6x - 30.$ (2)

Transposing, etc., $-x = -7;$ (3)

$$x = 7.$$

35.
$$\frac{x}{2}+3=\frac{x}{4}-\frac{x-2}{5}. \tag{1}$$

Clearing, $10x+60=5x-4x+8.$ (2)

Transposing, etc., $9x=-52;$ (3)

$$x=-5\tfrac{7}{9}.$$

36.
$$\frac{x-1}{2}+\frac{x-3}{4}-\frac{x-2}{3}=\frac{2}{3}. \tag{1}$$

Clearing, $6x-6+3x-9-4x+8=8.$ (2)

Transposing, etc., $5x=15;$ (3)

$$x=3.$$

37.
$$\frac{1-2x}{3}-\frac{4-5x}{4}=-\frac{13}{42}. \tag{1}$$

Clearing, $28-56x-84+105x=-26.$ (2)

Transposing, etc., $49x=30;$ (3)

$$x=\tfrac{30}{49}.$$

38.
$$\frac{x+3}{2}-\frac{x-2}{3}-\frac{1}{4}=\frac{3x-5}{12}. \tag{1}$$

Clearing, $6x+18-4x+8-3=3x-5.$ (2)

Transposing, etc., $-x=-28;$ (3)

$$x=28.$$

39.
$$\frac{4x-2}{11}+4-\frac{3x-5}{13}=5. \tag{1}$$

Clearing, $52x-26+572-33x+55=715.$ (2)

Transposing, etc., $19x=114;$ (3)

$$x=6.$$

40.
$$\frac{3x-3}{4}-\frac{3x-3}{3}=\frac{15}{3}-\frac{27+4x}{9}. \tag{1}$$

Clearing, $27x-27-36x+36=180-108-16x.$ (2)

Transposing, etc., $7x=63;$ (3)

$$x=9.$$

Page 113.

50. Let x represent his salary;

then $\frac{x}{3}=$ the sum paid for board;

$\frac{2x}{9}=$ the sum paid for other expenses.

Therefore, $\frac{x}{3}+\frac{2x}{9}+280=x.$ (1)

Clearing, $3x+2x+2520=9x.$ (2)

Transposing, $-4x=-2520;$ (3)

$x=630$, his salary.

Page 114.

51. Lex x represent the number;

then, $4\left(50-\frac{x}{5}\right)=x-70.$ (1)

$200-\frac{4x}{5}=x-70.$ (2)

Clearing, $1000-4x=5x-350;$ (3)

$-9x=-1350;$ (4)

$x=150$, the number.

52. Let x represent one part;

then, $100-x=$ the other part.

Therefore, $\frac{100-x}{4}-\frac{x}{3}=11.$ (1)

Clearing, $300-3x-4x=132;$ (2)

$-7x=-168;$ (3)

$x=24$, one part;

$100-x=76$, other part.

53. Let x represent the first;

then, $x+1=$ the second.

Therefore, $\frac{x}{2}+\frac{x}{5}=\frac{x+1}{3}+\frac{x+1}{4}.$

Clearing, $30x+12x=20x+20+15x+15;$

$7x=35;$

$x=5$, the first;

$x+1=6$, the second.

54. Let x represent A's age now;

then, $\frac{x-5}{2\frac{1}{3}}$, or $\frac{3x-15}{7}=$ B's age 5 years ago.

Hence, $\frac{3x-15}{7}+5=$ B's age now;

also, $\frac{x+1}{1\frac{4}{9}}$, or $\frac{9x+9}{13} =$ B's age 1 year hence.

Hence, $\frac{9x+9}{13} - 1 =$ B's age now.

Therefore, $\frac{9x+9}{13} - 1 = \frac{3x-15}{7} + 5.$ (1)

Clearing, $63x + 63 - 91 = 39x - 195 + 455.$ (2)

Transposing, etc., $24x = 288;$ (3)

$x = 12$, A's age now;

$\frac{3x-15}{7} + 5 = 8$, B's age now.

55. Let x represent the smaller;

then, $x + 20 =$ the larger.

Hence, $\frac{x+20}{7} = \frac{x}{3}.$ (1)

Clearing, $3x + 60 = 7x.$ (2)

Transposing, etc., $x = 15$, the smaller;

$x + 20 = 35$, the larger.

56. Let x represent the number;

then, $33 - \frac{x}{4} - \frac{x}{5} - \frac{x}{10} = 0.$

Clearing, $660 - 5x - 4x - 2x = 0.$

Transposing, etc., $x = 60.$

57. Let x represent the less;

then, $x + 8 =$ the greater.

Hence, $\frac{x+8}{x} = 3.$ (1)

Clearing, $x + 8 = 3x;$ (2)

$x = 4$, the less;

$x + 8 = 12$, the greater.

ADDITION AND SUBTRACTION OF FRACTIONS.

Page 116.

19. $\frac{y^2 - 2xy - x^2}{x^2 - xy} + \frac{x}{x-y} = \frac{y^2 - 2xy - x^2}{x^2 - xy} + \frac{x^2}{x^2 - xy} = \frac{y^2 - 2xy}{x^2 - xy}.$

20. $\frac{1}{2(x-1)} - \frac{1}{2(x+1)} + \frac{1}{x^2} = \frac{x^3 + x^2}{2x^2(x^2-1)} - \frac{x^3 - x^2}{2x^2(x^2-1)} + \frac{2x^2 - 2}{2x^2(x^2-1)}$

$= \frac{2x^2 - 1}{x^2(x^2-1)}.$

21. $\frac{1+x}{1+x+x^2}+\frac{1-x}{1-x+x^2}=\frac{1+x^3}{1+x^2+x^4}+\frac{1-x^3}{1+x^2+x^4}=\frac{2}{1+x^2+x^4}.$

Page 117.

30. $\frac{x}{1-x}-\frac{x^2}{1-x^2}+\frac{x}{1+x^2}=\frac{x+\cancel{x^2}+\cancel{x^3}+x^4}{1\cancel{+}x^4}-\frac{\cancel{x^2}+x^4}{1-x^4}+\frac{x-\cancel{x^3}}{1-x^4}=\frac{2x}{1-x^4}.$

31. $\frac{1}{(x-a)}+\frac{4a}{(x-a)^2}-\frac{5a^2}{(x-a)^3}=\frac{x^2-2ax+a^2}{(x-a)^3}+\frac{4ax-4a^2}{(x-a)^3}-\frac{5a^2}{(x-a)^3}$

$=\frac{x^2+2ax-8a^2}{(x-a)^3}=\frac{x^2+2ax-8a^2}{x^3-3ax^2+3a^2x-a^3}.$

32. $\frac{3}{x}-\frac{5}{2x-1}-\frac{2x-7}{4x^2-1}=\frac{12x^2-3}{4x^3-x}-\frac{10x^2+5x}{4x^3-x}-\frac{2x^2-7x}{4x^3-x}=\frac{2x-3}{4x^3-x}.$

33. $\frac{x^2+y^2}{xy}-\frac{x^2}{xy+y^2}-\frac{y^2}{x^2+xy}=\frac{x^3+x^2y+xy^2+y^3}{x^2y+xy^2}-\frac{x^3}{x^2y+xy^2}-\frac{y^3}{x^2y+xy^2}$

$=\frac{x^2y+xy^2}{x^2y+xy^2}=1.$

36. $\frac{3}{2x^2-2x}+\frac{6}{3x^2+6x}=\frac{9x+18}{6x(x^2+x-2)}+\frac{12x-12}{6x(x^2+x-2)}=\frac{7x+2}{2x(x^2+x-2)}.$

Page 118.

37. $\frac{x-3}{x^2-1}+\frac{4}{x+2}-\frac{5}{1-x}=\frac{x-3}{x^2-1}+\frac{4}{x+2}+\frac{5}{x-1}$

$=\frac{x^2-x-6}{(x+2)(x^2-1)}+\frac{4x^2-4}{(x+2)(x^2-1)}+\frac{5x^2+15x+10}{(x+2)(x^2-1)}=\frac{2x(5x+7)}{x^3+2x^2-x-2}.$

38. $\frac{a+1}{a^2+a+1}+\frac{1-a}{a^2-a+1}=\frac{a^3+1}{a^4+a^2+1}+\frac{-a^3+1}{a^4+a^2+1}=\frac{2}{a^4+a^2+1}.$

39. $\frac{1}{x^2-y^2}-\frac{1}{x^2+y^2}+\frac{2y^2}{x^4-y^4}=\frac{x^2+y^2}{x^4-y^4}-\frac{x^2-y^2}{x^4-y^4}+\frac{2y^2}{x^4-y^4}=\frac{4y^2}{x^4-y^4}.$

40. $\frac{a-b}{a^2+ab+b^2}-\frac{2}{b-a}+\frac{a^2-3b^2}{a^3+b^3}=\frac{a-b}{a^2+ab+b^2}+\frac{2}{a-b}+\frac{a^2-3b^2}{a^3+b^3}$

$=\frac{(a-b)^2(a^3+b^3)}{a^6-b^6}+\frac{2(a^3+b^3)(a^2+ab+b^2)}{a^6-b^6}+\frac{(a^2-3b^2)(a^3-b^3)}{a^6-b^6}$

$=\frac{4a^5+2a^2b^3+6b^5}{a^6-b^6}.$

41. $\frac{x^3+2x^2y}{x^4-y^4}+\frac{x-y}{x^2+y^2}-\frac{1}{x+y}$

$$=\frac{x^3+2x^2y}{x^4-y^4}+\frac{x^3-x^2y-xy^2+y^3}{x^4-y^4}-\frac{x^3-x^2y+xy^2-y^3}{x^4-y^4}.$$

$$=\frac{x^3+2x^2y-2xy^2+2y^3}{x^4-y^4}.$$

42. $\left(2-\frac{4}{x+2}\right)-\left(x+4-\frac{3x}{x-1}\right)$; $2-(x+4)=-2-x$;

$$\frac{-4}{x+2}+\frac{3x}{x-1}=-\frac{4x-4}{x^2+x-2}+\frac{3x^2+6x}{x^2+x-2}=\frac{3x^2+2x+4}{x^2+x-2}.$$

Therefore, the result is $\frac{3x^2+2x+4}{x^2+x-2}-x-2=\frac{10-x}{x^2+x-2}-x+1.$

43. $\frac{x+2}{x^2-9x+18}-\frac{x-1}{x^2+x-12}=\frac{x^2+6x+8}{x^3-5x^2-18x+72}-\frac{x^2-7x+6}{x^3-5x^2-18x+72}$

$$=\frac{13x+2}{x^3-5x^2-18x+72}.$$

44. $\left(m-\frac{4n-p}{3}\right)-\left(4m-\frac{2n+3p}{2}\right)$; $m-4m=-3m$;

$$-\frac{4n-p}{3}+\frac{2n+3p}{2}=-\frac{8n-2p}{6}+\frac{6n+9p}{6}=\frac{11p-2n}{6};$$

Therefore, the entire answer is $\frac{11p-2n}{6}-3m.$

45. $\frac{x+1}{x^2+x+1}-\frac{x-1}{x^2-x+1}=\frac{x^3+1}{x^4+x^2+1}-\frac{x^3-1}{x^4+x^2+1}=\frac{2}{x^4+x^2+1}.$

47. $\frac{a+2}{a^2-2a-3}-\frac{a+1}{a^2-a-6}=\frac{a^2+4a+4}{a^3-7a-6}-\frac{a^2+2a+1}{a^3-7a-6}=\frac{2a+3}{a^3-7a-6}.$

48. $\frac{3m}{m^2-5m-14}-\frac{2m}{m^2-3m-10}$

$$=\frac{3m^2-15m}{(m-7)(m-5)(m+2)}-\frac{2m^2-14m}{(m-7)(m-5)(m+2)}$$

$$=\frac{m(m-1)}{m^3-10m^2+11m+70}.$$

49. $\frac{1}{x+2}-\frac{9x-13-x^2}{x^2+5x+6}=\frac{x+3}{x^2+5x+6}-\frac{9x-13-x^2}{x^2+5x+6}=\frac{(x-4)^2}{x^2+5x+6}.$

50. $$\frac{x^2-y}{x^2-y^2}-\frac{x^2+y}{x^2+y^2}=\frac{x^4-x^2y+x^2y^2-y^3}{x^4-y^4}-\frac{x^4+x^2y-x^2y^2-y^3}{x^4-y^4}$$
$$=\frac{2x^2y(y-1)}{x^4-y^4}.$$

51. $$\frac{(x^2+y^2)^2}{(x-y)^2xy}-\left(\frac{y}{x}+2+\frac{x}{y}\right)=\frac{x^4+2x^2y^2+y^4}{(x-y)^2xy}-\frac{x^4-2x^2y^2+y^4}{(x-y)^2xy}$$
$$=\frac{4x^2y^2}{(x-y)^2xy}=\frac{4xy}{x^2-2xy+y^2}.$$

52. $$\frac{x^2+x-1}{2x^2-x-3}-\frac{x^2-x-1}{3x^2-x-4}$$
$$=\frac{3x^3-x^2-7x+4}{(2x-3)(x+1)(3x-4)}-\frac{2x^3-5x^2+x+3}{(2x-3)(x+1)(3x-4)}$$
$$=\frac{x^3+4x^2-8x+1}{6x^3-11x^2-5x+12}.$$

Page 119.

53. $$\frac{x-2}{x^2+4x-21}+\frac{2-x}{x^2+2x-15}+\frac{2}{x^2-12x+35}$$
$$=\frac{x^2+3x-10}{(x+7)(x-3)(x+5)}+\frac{-x^2-5x+14}{(x-7)(x-3)(x+5)}+\frac{2x-6}{(x+7)(x-3)(x+5)}$$
$$=\frac{-2}{x^3+9x^2-x-105}.$$

54. $$\frac{x-2}{2x^3-6x^2-4x+12}+\frac{x+6}{4x^3+2x^2-8x-4}$$
$$=\frac{2x^2-3x-2}{4x^4-10x^3-14x^2+20x+12}+\frac{x^2+3x-18}{4x^4-10x^3-14x^2+20x+12}$$
$$=\frac{3x^2-20}{4x^4-10x^3-14x^2+20x+12}.$$

55. $$\frac{1}{x+y}+\frac{1}{x-y}+\frac{x-y}{(x+y)^2}+\frac{x}{x^2-y^2}+\frac{x+y}{(x+y)^2}$$
$$=\frac{x^2-y^2}{(x+y)^2(x-y)}+\frac{x^2+2xy+y^2}{(x+y)^2(x-y)}+\frac{x^2-2xy+y^2}{(x+y^2)(x-y)}+\frac{x^2+xy}{(x+y)^2(x-y)}$$
$$+\frac{x^2-y^2}{(x+y)^2(x-y)}=\frac{5x^2+xy}{(x+y)^2(x-y)}=\frac{x(5x+y)}{x^3+x^2y-xy^2-y^3}.$$

MULTIPLICATION OF FRACTIONS.

Page 121.

18. $\dfrac{x+y}{(x-y)^2}\times\dfrac{x-y}{(x+y)^2}\times\dfrac{x-y}{x+y}=\dfrac{x+y}{(x-y)(x-y)}\times\dfrac{x-y}{(x+y)(x+y)}\times\dfrac{x-y}{x+y}$

$=\dfrac{1}{(x+y)^2}.$

19. $\dfrac{x^2+4x}{x^2-3x}\times\dfrac{6x^2-18x}{4x^2+16x}=\dfrac{x^2+4x}{x^2-3x}\times\dfrac{6(x^2-3x)}{4(x^2+4x)}=\dfrac{3}{2}=1\tfrac{1}{2}.$

20. $\dfrac{x^2-11x+30}{x^2-6x+9}\times\dfrac{x^2-3x}{x^2-5x}=\dfrac{(x-6)(x-5)}{(x-3)(x-3)}\times\dfrac{x(x-3)}{x(x-5)}=\dfrac{x-6}{x-3}.$

21. $\dfrac{x^2+x-2}{x^2-7x}\times\dfrac{x^2-13x+42}{x^2+2x}=\dfrac{(x+2)(x-1)}{x(x-7)}\times\dfrac{(x-7)(x-6)}{x(x+2)}=\dfrac{(x-1)(x-6)}{x^2}$

$=\dfrac{x^2-7x+6}{x^2}.$

22. $\dfrac{x^2+3x+2}{x^2-5x+6}\times\dfrac{x^2-7x+12}{x^2+x}=\dfrac{(x+2)(x+1)}{(x-3)(x-2)}\times\dfrac{(x-4)(x-3)}{x(x+1)}=\dfrac{(x+2)(x-4)}{x(x-2)}$

$=\dfrac{x^2-2x-8}{x^2-2x}.$

23. $\left(x+\dfrac{xy}{x-y}\right)\times\left(y-\dfrac{xy}{x+y}\right)=\dfrac{x^2-xy+xy}{x-y}\times\dfrac{xy+y^2-xy}{x+y}=\dfrac{x^2}{x-y}\times\dfrac{y^2}{x+y}$

$=\dfrac{x^2y^2}{x^2-y^2}.$

24. $\left(4+\dfrac{2x}{3c}\right)\times\left(2-\dfrac{2x}{6c}\right)=\dfrac{12c+2x}{3c}\times\dfrac{12c-2x}{6c}=\dfrac{72c^2-2x^2}{9c^2}.$

25. $\dfrac{y^2-2y+1}{y^2-4}\times\dfrac{y-2}{y^2+9y+10}=\dfrac{(y-1)(y-1)}{(y-2)(y+2)}\times\dfrac{y-2}{y^2+9y+10}$

$=\dfrac{y^2-2y+1}{y^3+11y^2+28y+20}.$

26. $(a^2-a+1)\left(\dfrac{1}{a^2}+\dfrac{1}{a}+1\right)=(a^2-a+1)\times\dfrac{1+a+a^2}{a^2}=\dfrac{a^4+a^2+1}{a^2}.$

27. $\dfrac{x^2-6x-16}{x^2+4x-21}\times\dfrac{x^2-8x+15}{x^2+9x+14}=\dfrac{(x-8)(x+2)}{(x+7)(x-3)}\times\dfrac{(x-5)(x-3)}{(x+7)(x+2)}$

$=\dfrac{x^2-13x+40}{x^2+14x+49}.$

28. $\left(\frac{x}{y}+1\right)\times\left(\frac{x}{y^2}-\frac{1}{y}+\frac{1}{x}\right)=\frac{x+y}{y}\times\frac{x^2-xy+y^2}{xy^2}=\frac{x^3+y^3}{xy^3}.$

29. $\frac{x^2+x-2}{x^2-3x}\times\frac{x^2+9x-36}{x^3+2x^2}=\frac{(x+2)(x-1)}{x(x-3)}\times\frac{(x-3)(x+12)}{x^2(x+2)}=\frac{x^2+11x-12}{x^3}.$

DIVISION OF FRACTIONS.

Page 123.

9. $\frac{5xy}{a-x}\div\frac{10xy}{a^2-x^2}=\frac{5xy}{a-x}\times\frac{(a+x)(a-x)}{10xy}=\frac{a+x}{2}.$

10. $\frac{2ax+x^2}{a^3-x^3}\div\frac{x}{a-x}=\frac{2ax+x^2}{(a-x)(a^2+ax+x^2)}\times\frac{a-x}{x}=\frac{2a+x}{a^2+ax+x^2}.$

11. $\frac{m^2-n^2}{6}\div\frac{3m+n}{12}=\frac{m^2-n^2}{6}\times\frac{12}{3m+n}=\frac{2m^2-2n^2}{3m+n}.$

12. $\frac{a^2-c^2}{1+x}\div(a+c)=\frac{(a-c)(a+c)}{1+x}\times\frac{1}{a+c}=\frac{a-c}{1+x}.$

13. $\frac{a^2+4x+4}{d+c}\div(a+z)=\frac{a^2+4x+4}{d+c}\times\frac{1}{a+z}=\frac{a^2+4x+4}{ad+ac+dz+cz}.$

14. $\frac{5(x+y)^2}{x-y}\div(x+y)=\frac{5(x+y)(x+y)}{x-y}\times\frac{1}{x+y}=\frac{5(x+y)}{x-y}.$

15. $\frac{ab+cd}{a+c}\div(a-c)=\frac{ab+cd}{a+c}\times\frac{1}{a-c}=\frac{ab+cd}{a^2-c^2}.$

16. $\frac{3an+cm}{x^2-y^2}\div(x^2+y^2)=\frac{3an+cm}{x^2-y^2}\times\frac{1}{x^2+y^2}=\frac{3an+cm}{x^4-y^4}.$

17. $\frac{4ax+4by}{c^2+d^2}\div(a+b)=\frac{4ax+4by}{c^2+d^2}\times\frac{1}{a+b}=\frac{4(ax+by)}{ac^2+ad^2+bc^2+bd^2}.$

18. $\frac{5xy+5xz}{a+n}\div5(x+z)=\frac{5(xy+xz)}{a+n}\times\frac{1}{5(x+z)}=\frac{x(y+z)}{ax+nx+az+nz}.$

19. $\frac{1}{x^2-17x+30}\div\frac{1}{x-15}=\frac{1}{(x-15)(x-2)}\times\frac{x-15}{1}=\frac{1}{x-2}.$

20. $\frac{a}{x^2-5x-6}\div\frac{b}{x^2+x}=\frac{a}{(x-6)(x+1)}\times\frac{x(x+1)}{b}=\frac{ax}{b(x-6)}.$

21. $\frac{b^2p+pxy}{a^2m}\div\frac{b^2c+cxy}{am^2}=\frac{p(b^2+xy)}{a^2m}\times\frac{am^2}{c(b^2+xy)}=\frac{mp}{ac}.$

22. $16\div\left(\frac{(a-c)^2}{a}-a+2c\right)=16\times\frac{a}{c^2}=\frac{16a}{c^2}.$

23. $$\frac{a^3-b^3}{a^2+ab+b^2}\div\frac{a-b}{a+b}=\frac{(a-b)(a^2+ab+b^2)}{a^2+ab+b^2}\times\frac{a+b}{a-b}=a+b.$$

24. $$\frac{a^2+b^2}{a^2+2ab+b^2}\div\frac{a-b}{a+b}=\frac{a^2+b^2}{(a+b)(a+b)}\times\frac{a+b}{a-b}=\frac{a^2+b^2}{a^2-b^2}.$$

25. $$\frac{a^2+ab+b^2}{a+b}\div\frac{a^2-ab+b^2}{a-b}=\frac{a^2+ab+b^2}{a+b}\times\frac{a-b}{a^2-ab+b^2}=\frac{a^3-b^3}{a^3+b^3}.$$

26. $$\frac{x^2+6x+8}{x^2-9}\div\frac{x^2-2x-8}{x-3}=\frac{(x+4)(x+2)}{(x-3)(x+3)}\times\frac{x-3}{(x-4)(x+2)}=\frac{x+4}{x^2-x-12}.$$

27. $$\frac{a^2-3a+2}{a^2-7a+10}\div\frac{a^2-1}{a^2-4a+4}=\frac{(a-2)(a-1)}{(a-5)(a-2)}\times\frac{(a-2)(a-2)}{(a-1)(a+1)}=\frac{a^2-4a+4}{a^2-4a-5}.$$

28. $$\frac{x^2-4x-12}{x^2+3x-28}\div\frac{x^2-3x-18}{x^2+6x-40}=\frac{(x-6)(x+2)}{(x+7)(x-4)}\times\frac{(x+10)(x-4)}{(x-6)(x+3)}$$
$$=\frac{x^2+12x+20}{x^2+10x+21}.$$

Page 124.

29. $$\frac{(x+y)^2-z^2}{z^2-(x-y)^2}\div\frac{z+y+x}{z-y+x}=\frac{(x+y+z)(x+y-z)}{(z+x-y)(z-x+y)}\times\frac{z-y+x}{z+x+y}$$
$$=\frac{x+y-z}{z+y-x}.$$

30. $$\frac{x^6-y^6}{x^4-y^4}\div\frac{(x^3-y^3)(x+y)}{x^2-y^2}=\frac{(x^3-y^3)(x^3+y^3)}{(x^2-y^2)(x^2+y^2)}\times\frac{x^2-y^2}{(x^3-y^3)(x+y)}$$
$$=\frac{x^2-xy+y^2}{x^2+y^2}.$$

31. $$\left(\frac{a^4-1}{a^3+1}\div\frac{a^2+1}{a+1}\right)\div\frac{a-1}{a^2-a+1}$$
$$=\frac{(a^2-1)(a^2+1)}{a^3+1}\times\frac{a+1}{a^2+1}\times\frac{a^2-a+1}{a-1}=a+1.$$

32. $$\left(\frac{m^2-7m-18}{n^2-11n+18}\div\frac{m^2-5m-14}{n^2-8n-9}\right)\div\frac{an+a}{6n-12}$$
$$=\frac{(m-9)(m+2)}{(n-9)(n-2)}\times\frac{(n-9)(n+1)}{(m-7)(m+2)}\times\frac{6(n-2)}{a(n+1)}=\frac{6(m-9)}{a(m-7)}.$$

Page 125.

14. $$\frac{x+\dfrac{y-x}{1+xy}}{1-\dfrac{xy-x^2}{1+xy}}=\frac{\dfrac{x^2y+y}{1+xy}}{\dfrac{1+x^2}{1+xy}}=\frac{x^2y+y}{1+xy}\times\frac{1+xy}{1+x^2}=\frac{y(x^2+1)}{1+x^2}=y.$$

15. $$\frac{a+5+\dfrac{1}{a-5}}{a-5+\dfrac{1}{a+5}}=\frac{a^2-24}{a-5}\times\frac{a+5}{a^2-24}=\frac{a+5}{a-5}.$$

16. $$\frac{4}{x+1}-\frac{3x-1}{x+\dfrac{x}{2}};\quad \frac{3x-1}{x+\dfrac{x}{2}}=(3x-1)\frac{2}{2x+x}=\frac{6x-2}{2x+x};\quad \frac{4}{x+1}-\frac{6x-2}{3x}$$

$$\frac{12x}{3x(x+1)}-\frac{6x^2+4x-2}{3x(x+1)}=\frac{2+8x-6x^2}{3x(x+1)}.$$

17. $$1-\frac{1}{1+\dfrac{1}{a}}=1-\frac{a}{a+1}=\frac{a+1-a}{a+1}=\frac{1}{a+1}.$$

18. $$\frac{1}{a+\dfrac{1}{a+\dfrac{1}{a}}}=\frac{1}{a+\dfrac{a}{a^2+1}}=1\times\frac{a^2+1}{a^3+2a}=\frac{a^2+1}{a(a^2+2)}.$$

19. $$\frac{\dfrac{x+y}{y}+\dfrac{y}{x+y}}{\dfrac{1}{x}+\dfrac{1}{y}}=\frac{x^2+2xy+2y^2}{y(x+y)}\times\frac{xy}{x+y}=\frac{x(x^2+2xy+2y^2)}{x^2+2xy+y^2}.$$

20. $$\frac{\left(\dfrac{a}{b}+\dfrac{b}{a}\right)\left(\dfrac{a}{b}-\dfrac{b}{a}\right)}{1-\dfrac{b-a}{b+a}}=\frac{\dfrac{a^2}{b^2}-\dfrac{b^2}{a^2}}{\dfrac{2a}{b+a}}=\frac{a^4-b^4}{a^2b^2}\times\frac{b+a}{2a}=\frac{a^5-ab^4+a^4b-b^5}{2a^3b^2}.$$

21. $$\frac{3x-2+\dfrac{1}{x}}{\dfrac{3x-1}{x}}=\frac{3x^2-2x+1}{x}\times\frac{x}{3x-1}=\frac{3x^2-2x+1}{3x-1}.$$

22. $$1+\frac{c}{1+c+\dfrac{2c^2}{1-c}}=1+\frac{c}{\dfrac{1+c^2}{1-c}}=1+\frac{c-c^2}{1+c^2}=\frac{1+c}{1+c^2}.$$

REVIEW OF FRACTIONS.

Page 125.

1. The highest common divisor of the numerator and denominator is x^2-3x+2. Dividing the terms of the fraction by this quantity, we have $\dfrac{x-3}{x+1}$.

2. The highest common divisor of the numerator and denominator is m^2+2m+3. Dividing the terms of the fraction by this quantity, we have $\dfrac{m-1}{m+1}$.

3. Dividing both terms of fraction by $x - 1$, the H. C. D. of numerator and denominator, we have $\frac{x^2 - x + 3}{x^2 - 4x + 9}$.

4. Dividing both terms of the fraction by $a^2 + a + 1$, H. C. D. of numerator and denominator, we have $\frac{a - 2}{a + 2}$.

5. Dividing both terms of fraction by $c + 3$, H. C. D. of numerator and denominator, we have $\frac{c + 5}{c^2 - 6c + 8}$.

6. The highest common divisor of the numerator and denominator is $x + 1$. Dividing the terms of the fraction by this quantity, we have

$$\frac{x^3 - 2x^2 - 2x + 1}{4x^2 - 7x - 1}.$$

7. The highest common divisor of the numerator and denominator is $a - 2$. Dividing the terms of the fraction by this quantity, we have

$$\frac{a^2 - 5a + 6}{3a^2 - 8a}.$$

8. Dividing both terms of fraction by $x^2 + 2x + 1$, H. C. D. of numerator and denominator, we have $\frac{x + 1}{3x - 7}$.

9. Dividing both terms of fraction by $2x - 1$, H. C. D. of numerator and denominator, we have $\frac{x^2 - x + 2}{x^3 + x + 1}$.

10. Dividing both terms of fraction by $a + 1$, H. C. D. of numerator and denominator, we have $\frac{2a^2 - 6a - 7}{6a^2 - 17a - 20}$.

11. The denominator of the second fraction may be changed to $x - 1$, by multiplying the terms of the fraction by -1. It becomes $-\frac{-x}{x - 1}$, or $+\frac{x}{x - 1}$, and the example may be written:

$$\frac{x}{x + 1} + \frac{x}{x - 1} + \frac{x^2}{x^2 - 1}.$$

The lowest common denominator of the fractions is $x^2 - 1$, and the fractions, when reduced, become:

$$\frac{x^2 - x}{x^2 - 1} + \frac{x^2 + x}{x^2 - 1} + \frac{x^2}{x^2 - 1} = \frac{3x^2}{x^2 - 1}.$$

Page 126.

12. The denominator of the third fraction may be changed to $4 - x^2$ by multiplying the terms of the fraction by -1. It becomes $\frac{x^2 - 16x}{4 - x^2}$, and the example may be written:

$$\frac{3 + 2x}{2 - x} - \frac{2 - 3x}{2 + x} + \frac{x^2 - 16x}{4 - x^2}.$$

The lowest common denominator of the fraction is $4 - x^2$, and the fractions, when reduced, become:

$$\frac{6 + 7x + 2x^2}{4 - x^2} - \frac{4 - 8x + 3x^2}{4 - x^2} + \frac{x^2 - 16x}{4 - x^2} = \frac{2 - x}{4 - x^2} = \frac{1}{2 + x}.$$

13. By multiplying the terms of the second fraction by -1, the fraction becomes $+\frac{-1}{(x - y)(x - z)}$, or $-\frac{1}{(x - y)(x - z)}$, and the fractions may be expressed as follows:

$$\frac{1}{(x - y)(y - z)} - \frac{1}{(x - y)(x - z)} + \frac{1}{(x - z)(x - y)}.$$

It will be observed that the second and third fractions are the same with opposite signs; hence the value of the expression is $\frac{1}{(x - y)(y - z)}$.

14. It will be observed that the lowest common denominator of the given fractions will be $abc(a - b)(a - c)(b - c)$, if the signs of the quantities in the denominators can be changed. Since they can be changed by multiplying the terms of the fractions by -1, we multiply the terms of the second fraction by -1, obtaining:

$$+\frac{-1}{b(a - b)(b - c)}, \text{ or } -\frac{1}{b(a - b)(b - c)}.$$

Multiplying the terms of the third fraction by -1, we have

$$+\frac{-1}{c(a - c)(c - b)}.$$

Multiplying this result again by -1, we have

$$+\frac{1}{c(a - c)(b - c)}.$$

The fractions may now be expressed as follows:

$$\frac{1}{a(a - b)(a - c)} - \frac{1}{b(a - b)(b - c)} + \frac{1}{c(a - c)(b - c)}.$$

Reducing them to equivalent fractions having a common denominator, we have the following:

$$\frac{bc(b-c)}{abc(a-b)(a-c)(b-c)}-\frac{ac(a-c)}{abc(a-b)(a-c)(b-c)}$$

$$+\frac{ab(a-b)}{abc(a-b)(a-c)(b-c)}=\frac{b^2c-bc^2-a^2c+ac^2+a^2b-ab^2}{abc(b^2c-bc^2-a^2c+ac^2+a^2b-ab^2)}=\frac{1}{abc}.$$

15. $1+\frac{x+y}{x-y}=\frac{x-y+x+y}{x-y}=\frac{2x}{x-y}$;

$$1-\frac{x-y}{x+y}=\frac{x+y-x+y}{x+y}=\frac{2y}{x+y};\quad \frac{2x}{x-y}\times\frac{2y}{x+y}=\frac{4xy}{x^2-y^2}.$$

16. $x-y+\frac{x^2+y^2}{x+y}=\frac{x^2-y^2+x^2+y^2}{x+y}=\frac{2x^2}{x+y}.$

Since the factors are alike, the product is $\frac{4x^4}{x^2+2xy+y^2}$.

17. $1-\frac{1}{a}+\frac{1}{a^2}=\frac{a^2-a+1}{a^2}$;

$$a^2+1+a\times\frac{a^2-a+1}{a^2}=\frac{a^4+a^2+1}{a^2}=a^2+1+\frac{1}{a^2}.$$

18. $\frac{a+1}{a-1}-\frac{a-1}{a+1}=\frac{a^2+2a+1}{a^2-1}-\frac{a^2-2a+1}{a^2-1}=\frac{4a}{a^2-1}.$

$$\frac{4a}{a^2-1}\div\frac{x}{a-1}=\frac{4a}{a^2-1}\times\frac{a-1}{x}=\frac{4a}{ax+x}.$$

19. $1+\frac{x-1}{x+1}=\frac{x+1+x-1}{x+1}=\frac{2x}{x+1}$;

$\frac{2x}{x-1}\times\frac{2x}{x+1}=\frac{4x^2}{x^2-1}$, the dividend simplified.

$$1-\frac{x-1}{x+1}=\frac{x+1-x+1}{x+1}=\frac{2}{x+1};$$

$$1+\frac{x+1}{x-1}=\frac{x-1+x+1}{x-1}=\frac{2x}{x-1};$$

$\frac{2}{x+1}\times\frac{2x}{x-1}=\frac{4x}{x^2-1}$, the divisor simplified.

$$\frac{4x^2}{x^2-1}\div\frac{4x}{x^2-1}=x.$$

20. $$\frac{1+\frac{a-x}{a+x}}{1-\frac{a-x}{a+x}}=\frac{\frac{a+x+a-x}{a+x}}{\frac{a+x-a+x}{a+x}}=\frac{\frac{2a}{a+x}}{\frac{2x}{a+x}}=\frac{2a}{2x}=\frac{a}{x};$$

$$\frac{1+\frac{a^2-x^2}{a^2+x^2}}{1-\frac{a^2-x^2}{a^2+x^2}}=\frac{\frac{a^2+x^2+a^2-x^2}{a^2+x^2}}{\frac{a^2+x^2-a^2+x^2}{a^2+x^2}}=\frac{\frac{2a^2}{a^2+x^2}}{\frac{2x^2}{a^2+x^2}}=\frac{2a^2}{2x^2}=\frac{a^2}{x^2};$$

$$\frac{a}{x}\div\frac{a^2}{x^2}=\frac{a}{x}\times\frac{x^2}{a^2}=\frac{x}{a}.$$

21. $$\frac{a-1}{6}-\frac{2a-7}{2}=\frac{20-5a}{6};\quad \frac{3a}{4}-3=\frac{3a-12}{4};\quad \frac{20-5a}{6}\times\frac{4}{3a-12}=-1\tfrac{1}{3}.$$

22. $$1-\cfrac{1}{1-\cfrac{1}{1-\cfrac{1}{x}}}=1-\cfrac{1}{1-\cfrac{x}{x-1}}=1-(1-x)=x.$$

23. $$\cfrac{4}{1+\cfrac{4}{1+\cfrac{4}{1+\cfrac{4}{x}}}}=\cfrac{4}{1+\cfrac{4}{1+\cfrac{4x}{x+4}}}=\cfrac{4}{1+\cfrac{4x+16}{5x+4}}=\frac{4(5x+4)}{9x+20}.$$

24. $$\frac{x^2+y^2}{x^2-y^2}-\frac{x^2-y^2}{x^2+y^2}=\frac{4x^2y^2}{x^4-y^4};\quad \frac{x+y}{x-y}-\frac{x-y}{x+y}=\frac{4xy}{x^2-y^2};\quad \frac{4x^2y^2}{x^4-y^4}\times\frac{x^2-y^2}{4xy}=\frac{xy}{x^2+y^2}.$$

Page 127.

25. $$x^2+xy+y^2+\frac{x^3+y^3}{x-y}=\frac{2x^3}{x-y};$$

$$x-\frac{(x^2+2y)^2}{x^2-y^2}-4=\frac{x^3-xy^2-x^4-4x^2y-4x^2}{x^2-y^2};$$

$$\frac{2x^3}{x-y}\times\frac{x^2-y^2}{x^3-xy^2-x^4-4xy^2-4x^2}=\frac{2x^3(x+y)}{x^3-xy^2-x^4-4x^2y-4x^2}.$$

26. $$\frac{x^2+2x}{x^2-8x+12}-\frac{3x^2-1}{3x^2-20x+12}=\frac{10x^2-3x-2}{(x-6)(x-2)(3x-2)};$$

$$\frac{3}{3x^2-8x+4}-\frac{1}{x^2+3x-10}=\frac{17}{(x+5)(x-2)(3x-2)};$$

$$\frac{10x^2-3x-2}{(x-6)(x-2)(3x-2)}\times\frac{(x+5)(x-2)(3x-2)}{17}=\frac{10x^3+47x^2-17x-10}{17(x-6)}.$$

27. $$\frac{1+\dfrac{a-c}{a+c}}{1-\dfrac{a-c}{a+c}}=\frac{2a}{a+c}\times\frac{a+c}{2c}=\frac{a}{c};$$

$$\frac{1+\dfrac{a^2-c^2}{a^2+c^2}}{1-\dfrac{a^2-c^2}{a^2+c^2}}=\frac{2a^2}{a^2+c^2}\times\frac{a^2+c^2}{2c^2}=\frac{a^2}{c^2};\ \frac{a}{c}\times\frac{c^2}{a^2}=\frac{c}{a}.$$

28. $$\frac{x-5-\dfrac{3x-15}{x+3}}{x-1}=\frac{x^2-5x}{x+3}\times\frac{1}{x-1}=\frac{x^2-5x}{(x+3)(x-1)};$$

$$\frac{x+4-\dfrac{x^2-13}{x-1}}{\dfrac{1}{5}-\dfrac{1}{x}}=\frac{3x+9}{x-1}\times\frac{5x}{x-5}=\frac{15x^2+45x}{(x-5)(x-1)};$$

$$\frac{x(x-5)}{(x+3)(x-1)}\times\frac{15x(x+3)}{(x-5)(x-1)}=\frac{15x^2}{(x-1)^2}.$$

29. $$1-\frac{\dfrac{x^2}{y^2}-\dfrac{y^2}{x^2}}{\dfrac{x^2}{y^2}+\dfrac{y^2}{x^2}}=1-\left(\frac{x^4-y^4}{x^2y^2}\times\frac{x^2y^2}{x^4+y^4}\right)=1-\frac{x^4-y^4}{x^4+y^4}=\frac{2y^4}{x^4+y^4};\ \frac{x^3}{y}+\frac{y^3}{x}$$

$$=\frac{x^4+y^4}{xy};\ \frac{2y^4}{x^4+y^4}\times\frac{x^4+y^4}{xy}=\frac{2y^3}{x}.$$

30. $$\frac{x^2y^2}{(x-y)^2+2xy}-y^2=\frac{-y^4}{x^2+y^2};\ \frac{y^4-x^4}{y^6}-\frac{2xy-2y^2}{y^2}=\frac{y^4-x^4}{y^6}-\frac{2xy^5-2y^6}{y^6}$$

$$=\frac{y^4-x^4-2xy^5+2y^6}{y^6};\ \frac{-y^4}{x^2+y^2}\times\frac{y^4-x^4-2xy^5+2y^6}{y^6}$$

$$=\frac{x^4+2xy^5-y^4-2y^6}{x^2y^2+y^4}.$$

31. $$4(a-b)^2\div\frac{ab}{a+b}=4(a-b)^2\times\frac{a+b}{ab}=\frac{4(a^2-b^2)(a-b)}{ab};$$

$$(a^2-b^2)(2a^2-2ab)\times\frac{ab}{4(a^2-b^2)(a-b)}=\frac{a^2b}{2}.$$

32. $$\frac{\dfrac{a-1}{a}-\dfrac{b+1}{b}-\dfrac{1}{c}}{\dfrac{a}{bc}-\dfrac{b}{ac}}=\frac{-bc-ac-ab}{abc}\times\frac{abc}{a^2-b^2}=\frac{-ab-bc-ac}{a^2-b^2};$$

$$\frac{b}{a+b}\times\frac{b^2}{a-b}\times\frac{1}{bc}=\frac{b^2}{c(a^2-b^2)};$$

$$\frac{-ab-bc-ac}{a^2-b^2}+\frac{b}{c(a+b)}+\frac{b^2}{c(a^2-b^2)}=\frac{ab-bc^2-ac^2-abc}{c(a^2-b^2)}.$$

33. $$\frac{x^2+4x-12}{y^2+2y-15}\times\frac{x(y+5)}{y(x+6)}=\frac{x(x-2)}{y(y-3)};$$

$$\frac{x^2-14x+24}{y^2+9y-36}\times\frac{y(y-3)}{x(x-2)}=\frac{y(x-12)}{x(y+12)}.$$

Page 128.

34. $$\frac{a-b}{a+b}\times\frac{a^5+b^5}{a^2+b^2}=\frac{(a-b)(a^4-a^3b+a^2b^2-ab^3+b^4)}{a^2+b^2};$$

$$\frac{(a^4-a^3b+a^2b^2-ab^3+b^4)}{a-b}\times\frac{a^2+b^2}{(a-b)(a^4-a^3b+a^2b^2-ab^3+b^4)}=\frac{a^2+b^2}{(a-b)^2}.$$

35. $$\frac{1+a}{1+a+a^2}+\frac{1-a}{1-a+a^2}=\frac{2}{1+a^2+a^4};$$

$$\frac{2}{1+a^2+a^4}\times\frac{2}{a}=\frac{4}{a(1+a^2+a^4)};$$

$$1\div\frac{4}{a(1+a^2+a^4)}=1\times\frac{a(1+a^2+a^4)}{4}=\frac{a+a^3+a^5}{4}.$$

36. $$1+\frac{x}{x-2}=\frac{2x-2}{x-2};\quad \frac{x^2-3x-4}{x^2+x-2}\times\frac{4-x^2}{1-x^2}\times\frac{2x-2}{x-2}=\frac{2(4-x)}{1-x};$$

$$\frac{2x-8}{x-1}-\frac{2x}{x-4}=\frac{32-14x}{x^2-5x+4}.$$

37. $$\frac{x-2}{x^2-4}+\frac{x-3}{x-2}-\frac{x}{x+2}=\frac{2x-8}{x^2-4};$$

$$\frac{2x-8}{x^2-4}\times\frac{x^2-4}{2}\times\frac{6x}{x^2-36}=\frac{6x(x-4)}{x^2-36}.$$

38. $$\frac{a+x}{x}-\frac{2a}{a+x}+\frac{a^2x-a^3}{x(a^2-x^2)}$$

$$=\frac{a^3-ax^2+a^2x-x^3}{x(a^2-x^2)}-\frac{2a^2x-2ax^2}{x(a^2-x^2)}+\frac{a^2x-\cancel{a^3}}{x(a^2-x^2)}$$

$$=\frac{ax^2-x^3}{x(a^2-x^2)}=\frac{x^2(a-x)}{x(a^2-x^2)}=\frac{x}{a+x}.$$

39. $$\frac{x+y}{xy}\times\frac{(x+y)(x-y)}{3(x^2+y^2)}\times\frac{3xy(x^2+y^2)}{(x-y)^2(x+y)^2}=\frac{1}{x-y}.$$

40. $$\frac{2x^2y}{x^3+y^3}-\frac{2xy^2}{x^3-y^3}+\frac{x^2+y^2}{x^2-y^2}$$
$$=\frac{2x^5y-2x^2y^4}{x^6-y^3}-\frac{2x^4y^2+2xy^5}{x^6-y^6}+\frac{x^6+2x^4y^2+2x^2y^4+y^6}{x^6-y^6}$$
$$=\frac{x^6+2x^5y-2xy^5+y^6}{x^6-y^6}.$$

41. $$\frac{ax}{ax+c}-\frac{c}{ax-c}+\frac{ax(3c-ax)}{a^2x^2-c^2}$$
$$=\frac{a^2x^2-acx}{a^2x^2-c^2}-\frac{acx+c^2}{a^2x^2-c^2}+\frac{3acx-a^2x^2}{a^2x^2-c^2}=\frac{acx-c^2}{a^2x^2-c^2}=\frac{c}{ax+c}.$$

42. $$\frac{3a+x}{x-3a}\times\frac{x}{a}=\frac{x(3a+x)}{a(x-3a)};\quad \frac{(3x+a)(x+3a)}{(x-3a)(a-3x)}\times\frac{a(x-3a)}{x(3a+x)}=\frac{a(3x+a)}{x(a-3x)}.$$

43. $$\frac{1}{b}-\frac{1}{a}=\frac{a-b}{ab};\quad \frac{a^2-b^2}{a^3+b^3}\times\frac{a^2+b^2-ab}{b}\times\frac{ab}{a-b}$$
$$=\frac{(a-b)(a+b)}{(a+b)(a^2-ab+b^2)}\times\frac{a^2+b^2-ab}{b}\times\frac{ab}{a-b}=a.$$

44. $$\frac{1-a^2}{1-a^3}+\frac{1-a}{1-a+a^2}=\frac{2(1-a)}{(1-a^3)(1-a+a^2)};\quad \frac{1+a}{1+a+a^2}-\frac{1-a^2}{1+a^3}$$
$$=\frac{2a^3(1+a)}{(1+a^3)(1+a+a^2)};\quad \frac{2(1-a)}{(1-a^3)(1-a+a^2)}\times\frac{(1+a^3)(1+a+a^2)}{2a^3(1+a)}$$
$$=\frac{2(1-a)}{(1-a)(1+a+a^2)(1-a+a^2)}\times\frac{(1+a)(1-a+a^2)(1+a+a^2)}{2a^3(1+a)}=\frac{1}{a^3}.$$

45. $$\frac{(a+x)^2-y^2}{a^2+ax-ay}\times\frac{a}{(a+y)^2-x^2}=\frac{(a+x+y)(a+x-y)}{a(a+x-y)}\times\frac{a}{(a+y+x)(a+y-x)}$$
$$=\frac{1}{a+y-x};\quad \frac{a-x^2-y^2}{x(a-x-y)}\times\frac{1}{a+y-x}=\frac{a-x^2-y^2}{x(a-x+y)(a-x-y)}.$$

46. $$\frac{(a-b)(a+b)}{ab^2x}\times\frac{b(a-b)}{(a+b)(a+b)}\times\frac{b(a+b)}{(a-b)(a-b)}=\frac{1}{ax}.$$

47. $$\frac{a-x}{a+x}-\frac{a^3-x^3}{a^3+x^3}=\frac{a^3-2a^2x-x^3}{a^3+x^3}-\frac{a^3-x^3}{a^3+x^3}=\frac{2ax^2-2a^2x}{a^3+x^3};$$
$$\frac{a+x}{a-x}+\frac{a^2+x^2}{a^2-x^2}=\frac{a^2+2ax+x^2}{a^2-x^2}+\frac{a^2+x^2}{a^2-x^2}=\frac{2a^2+2ax+2x^2}{a^2-x^2};$$
$$\frac{2ax^2-2a^2x}{a^3+x^3}\times\frac{a^2-x^2}{2(a^2+ax+x^2)}=\frac{2ax(x-a)}{(a+x)(a^2-ax+x^2)}\times\frac{(a+x)(a-x)}{2(a^2+ax+x^2)}$$
$$=\frac{ax(2ax-x^2-a^2)}{a^4+a^2x^2+x^4}.$$

SIMPLE EQUATIONS.

Page 130.

1. Clearing, $8x - x - 2 = 6x + 6.$ (1)
Transposing, etc., $x = 8.$ (2)

2. Clearing, $18x - 18x - 24 = 2x + 3x - 36.$ (1)
Transposing, etc., $5x = 12;$ (2)
whence, $x = 2\frac{2}{5}.$ (3)

3. Clearing, $12x - 16 + 8 = 4x - 5 + 2x.$ (1)
Transposing, etc., $6x = 3;$ (2)
whence, $x = \frac{1}{2}.$ (3)

4. Clearing, $24x - 72 - 3x - 6 = 8x.$ (1)
Transposing, etc., $13x = 78;$ (2)
whence, $x = 6.$ (3)

5. Clearing, $15x = 9 - 6 + 2x.$ (1)
Transposing, etc., $13x = 3;$ (2)
whence, $x = \frac{3}{13}.$ (3)

6. Clearing, $11x - 33x = 3x - 3 - 297.$ (1)
Transposing, etc., $25x = 300;$ (2)
whence, $x = 12.$ (3)

7. Clearing, $45x - 7x - 21 = 70x - 735.$ (1)
Transposing, etc., $32x = 714;$ (2)
whence, $x = 22\frac{5}{16}.$ (3)

8. Clearing, $ax - b + ac = x + ac.$ (1)
Transposing, etc., $ax - x = b;$ (2)
or, $(a - 1)x = b;$ (3)
whence, $x = \frac{b}{a - 1}.$ (4)

9. Clearing, $4ax - 6a + 2bx = 1.$ (1)
Transposing, etc., $4ax + 2bx = 1 + 6a;$ (2)
or, $(4a + 2b)x = 1 + 6a;$ (3)
whence, $x = \frac{1 + 6a}{4a + 2b}.$ (4)

10. Clearing, $dx - abd = ac - adx.$ (1)

Transposing, etc., $dx + adx = ac + abd;$ (2)

or, $(d + ad)x = ac + abd;$ (3)

whence, $x = \frac{ac + abd}{d + ad}.$ (4)

11. Clearing, $9x - 15 - 72 = 8 - 4x - 6x;$ (1)

Transposing, etc., $19x = 95;$ (2)

whence, $x = 5.$ (3)

12. Clearing, $ax + x - ax + x = a^2b - b.$ (1)

Uniting, etc., $2x = a^2b - b;$ (2)

whence, $x = \frac{a^2b - b}{2}.$ (3)

13. Simplifying terms, $x + a = \frac{a}{4}.$ (1)

Transposing, etc., $x = -\frac{3a}{4}.$ (2)

14. Clearing, $24 - 24x = 3x + 24 - 4x - 24.$ (1)

Transposing, etc., $23x = 24;$ (2)

whence, $x = 1\frac{1}{23}.$ (3)

15. Clearing, $24x + 45b = 5a + 180b.$ (1)

Transposing, etc., $24x = 5a + 135b;$ (2)

whence, $x = \frac{5a + 135b}{24}.$ (3)

16. Clearing, $c^2x - acx - a^2c^2 + a^3c = a^2c - a^3 - acx.$ (1)

Transposing, etc., $c^2x = a^2c^2 - a^3c + a^2c - a^3.$ (2)

Factoring, $c^2x = a^2c(c - a) + a^2(c - a).$ (3)

Uniting terms, $c^2x = (a^2c + a^2)(c - a).$ (4)

Factoring, $c^2x = a^2(c + 1)(c - a);$ (5)

whence, $x = \frac{a^2(c + 1)(c - a)}{c^2}.$ (6)

17. Clearing, $6x - 24 + 4x = 360 - 6x - 192.$ (1)

Transposing, etc., $16x = 192;$ (2)

whence, $x = 12.$ (3)

18. Clearing, $30 - 3x - 4 = 6x - 10.$ (1)

Transposing, etc., $9x = 36;$ (2)

whence $x = 4.$ (3)

19. Simplifying, $9 + 10x - 6 + 2x = 27.$ (1)

Transposing, etc., $12x = 24;$ (2)

whence, $x = 2.$ (3)

20. Clearing, $30x - 10x + 20 + 6x - 24 = 210 + 5x - 25.$ (1)

Transposing, etc., $21x = 189;$ (2)

whence, $x = 9.$ (3)

21. Clearing, $3x^2 - 7x - 3x^2 + 15x = 6x - 14.$ (1)

Transposing, etc., $2x = -14;$ (2)

whence, $x = -7.$ (3)

22. It will be observed that the least common denominator of the first two fractions is $x^2 - 1$, and that the denominator of the fraction in the second member is $1 - x^2$, which are the same quantities with opposite signs. In such cases the denominators may be made the same by multiplying the terms of the fractions by -1. Multiplying the terms of the fraction in the second member by -1, and clearing of fractions, we have

$3x - 3 - x^2 - 2x - 1 = -x^2;$ (1)

whence, $x = 4.$ (2)

23. Clearing,

$2x^4 + 2x^3 - 9x^2 + 12 = 2x^4 + 2x^3 - 23x^2 + 7x + 12.$ (1)

Transposing, etc., $14x^2 = 7x;$ (2)

whence, $x = \frac{1}{2}.$ (3)

Page 131.

24. Clearing, $24 + 27x - 27 - 36x + 48 = 216 - 108 - 16x.$ (1)

Transposing, etc., $7x = 63;$ (2)

whence, $x = 9.$ (3)

25. The first fraction is equal to $\frac{x}{4} + \frac{20}{36}.$ Hence, the first member reduces to $\frac{20}{36}$, or $\frac{5}{9}.$

Clearing, $25x - 20 = 36x - 108;$ (1)

whence, $x = 8.$ (2)

26. The complex fraction in the second member reduces to $\frac{x}{6} - \frac{4x - 9}{18}.$

Clearing, $36x + 144 + 45x - 720 = 15x - 20x + 45 - 795;$ (1)

whence, $x = -2\frac{1}{43}.$ (2)

108
48
156

27
216
243
68

24
27
216
227
156

27. Clearing, $ax^2 - a^2x - 2abx + a^2b + ab^2 + bx^2 - b^2x$

$$= \cancel{ax^2} - \cancel{abx} - acx + \cancel{abc} + bx^2 - \cancel{abx} - bcx + \cancel{abc} \qquad (1)$$

Transposing, etc., $acx + bcx - a^2x - b^2x = 2abc - a^2b - ab^2$; (2)

factoring, $c(a+b)x - (a^2+b^2)x = ab(2c - a - b)$; (3)

whence, $$x = \frac{ab(2c-a-b)}{c(a+b)-(a^2+b^2)}. \qquad (4)$$

24
48
108
18

28. The first fraction in the first member is equal to $\frac{2x}{5} + \frac{13}{15}$, and the second member is equal to $\frac{2x}{5} + 3$.

Therefore, simplifying before clearing of fractions, we have

$$\frac{13}{15} = \frac{9x+15}{5x-25}. \qquad (1)$$

Clearing, $65x - 325 = 135x + 225$; (2)

whence, $x = -7\frac{6}{7}$. (3)

29. Clearing,

$$ax + bx - a^2 - ab - ax - a^2 + bx + ab = 2ax. \qquad (1)$$

Transposing, etc., $2bx - 2ax = 2a^2$; (2)

whence, $$x = \frac{a^2}{b-a}. \qquad (3)$$

31. Substituting y for $x + 6$, we have

$$y - \frac{3y}{4} = \frac{y}{3} - 6. \qquad (1)$$

Clearing, $12y - 9y = 4y - 72$; (2)

whence, $y = 72$. (3)

And, $x + 6 = 72$; (4)

whence, $x = 66$. (5)

32. Substituting y for $x - 7$, we have

$$y + \frac{y}{2} + \frac{3y}{4} = \frac{9}{4}. \qquad (1)$$

Clearing, etc., $y = 1$. (2)

And, $x - 7 = 1$; (3)

whence, $x = 8$. (4)

33. Substituting y for $x + 4$, we have

$$\frac{3y}{2} + \frac{y}{4} - \frac{3y}{5} = \frac{23}{2}. \qquad (1)$$

Clearing, etc., $y = 10.$ (2)

And, $x + 4 = 10;$ (3)

whence, $x = 6.$ (4)

34. Substituting y for $x - 3$, we have

$$\frac{2y}{3} + 2y = 5 - \frac{y}{4}. \qquad (1)$$

Clearing, etc., $y = 1\frac{5}{7}.$ (2)

And, $x - 3 = 1\frac{5}{7};$ (3)

whence, $x = 4\frac{5}{7}.$ (4)

Page 132.

35. The first fraction of the first member is equal to $7 - x$, whence, simplifying, we have

$$1 - x - \frac{2(2x + 3)}{9} = -\frac{5x + 1}{4}. \qquad (1)$$

Clearing, $36 - 36x - 16x - 24 = -45x - 9.$ (2)

Transposing, etc., $x = 3.$ (3)

36. Clearing, $6x - 24 - 9x + 15 = 48 - 4x + 8.$ (1)

Transposing, etc., $x = 65.$ (2)

37. The equation may be expressed:

$$\frac{4x}{9} + \frac{3}{9} + \frac{7x - 29}{5x - 12} = \frac{8x}{18} + \frac{19}{18}. \qquad (1)$$

Simplifying, $$\frac{7x - 29}{5x - 12} = \frac{13}{18}. \qquad (2)$$

Clearing, $126x - 522 = 65x - 156;$ (3)

whence, $x = 6.$

38. The equation may be expressed:

$$\frac{9x}{14} + \frac{5}{14} + \frac{8x - 7}{6x + 2} = \frac{36x}{56} + \frac{15}{56} + \frac{41}{56}. \qquad (1)$$

Simplifying, $$\frac{8x - 7}{6x + 2} = \frac{9}{14}. \qquad (2)$$

Clearing, $112x - 98 = 54x + 18;$ (3)

whence, $x = 2.$

39. The equation may be expressed:

$$\frac{6x}{15}+\frac{7}{15}-\frac{2x-2}{7x-6}=\frac{2x}{5}+\frac{1}{5}. \tag{1}$$

Simplifying,
$$\frac{2x-2}{7x-6}=\frac{4}{15}. \tag{2}$$

Clearing,
$$30x-30=24x-24; \tag{3}$$

whence,
$$x=3.$$

40. The equation may be expressed:

$$\frac{6x}{15}+\frac{1}{15}-\frac{2x-4}{7x-16}=\frac{2x}{5}-\frac{1}{5}. \tag{1}$$

Simplifying,
$$\frac{2x-4}{7x-16}=\frac{4}{15}. \tag{2}$$

Clearing,
$$30x-60=28x-64; \tag{3}$$

whence,
$$x=-2.$$

41. The equation may be written:

$$\frac{6x}{15}+\frac{13}{15}-\frac{3x+5}{5x-25}=\frac{2x}{5}. \tag{1}$$

Simplifying,
$$\frac{3x+5}{5x-25}=\frac{13}{15}. \tag{2}$$

Clearing,
$$45x+75=65x-325, \tag{3}$$

whence,
$$x=20.$$

42. Dividing numerator by denominator in first two fractions:

$$1-\frac{5}{x-7}+1+\frac{8}{x-12}=2+\frac{7}{x-7}. \tag{1}$$

Simplifying,
$$\frac{8}{x-12}=\frac{12}{x-7}. \tag{2}$$

Clearing,
$$8x-56=12x-144; \tag{3}$$

whence,
$$x=22.$$

43. The equation may be expressed:

$$\frac{2x}{4}-\frac{10}{4}-\frac{x}{5}+\frac{8}{5}-\frac{x}{2}+\frac{5}{2}=x-14. \tag{1}$$

Simplifying,
$$-\frac{x}{5}=x-\frac{78}{5}. \tag{2}$$

Clearing,
$$-x=5x-78; \tag{3}$$

whence,
$$x=13.$$

44. Clearing, $8x - 20 - 3x + 9 = 12x - 24 - 6x + 6.$ (1)

Transposing, etc., $x = 7.$ (2)

45. Clearing, $6x - 20x + 140 = 130 - 15x.$ (1)

Transposing, etc., $x = -10.$ (2)

46. Clearing, $3x - 6x + 22 = 24x - 20x - 13.$ (1)

Transposing, etc., $x = 5.$ (2)

47. The fraction in the second member of the equation is equal to $9 - \frac{27}{3x - 1}$. Transposing and uniting terms, we have

$$\frac{7x + 8}{3x - 1} = 21 - \frac{27}{3x - 1}. \quad (1)$$

Clearing, etc., $x = 1.$ (2)

48. Clearing, $27x - 27 - 36x + 48 = 192 - 108 - 16x.$ (1)

Transposing, etc., $x = 9.$ (2)

Page 133.

49. Clearing, $12x + 96 - 11 + 3x = 180x - 1554 - 39 + 3x - 42 + 4x.$ (1)

Transposing, etc., $172x = 1720;$ (2)

whence, $x = 10.$ (3)

50. The equation may be expressed:

$$\frac{4x}{9} + \frac{3}{9} = \frac{8x}{18} + \frac{19}{18} - \frac{7x - 29}{5x - 12}. \quad (1)$$

Simplifying, $\frac{7x - 29}{5x - 12} = \frac{13}{18}.$ (2)

Clearing, $126x - 522 = 65x - 156;$ (3)

whence, $x = 6.$

51. The equation may be expressed:

$$\frac{a}{b} + \frac{x}{b^2} - \frac{b}{a^2} + \frac{x}{a^2b} = \frac{x}{a^2} - \frac{b}{a^2} - \frac{a}{b} + \frac{x}{b^2}. \quad (1)$$

Simplifying, $\frac{x}{a^2b} = \frac{x}{a^2} - \frac{2a}{b}.$ (2)

Clearing, $x = bx - 2a^3;$ (3)

whence, $x = \frac{2a^3}{b - 1}.$

52. Clearing, $ab^2 + b^2x + abx + bx^2 - ab^2 - abc = a^2c + bx^2.$ (1)

Transposing, etc., $abx + b^2x = a^2c + abc.$ (2)

Factoring, $b(a + b)x = ac(a + b)$; (3)

whence, $x = \frac{ac}{b}.$ (4)

53. The equation may be expressed:

$$\frac{4x}{8} - 1 + \frac{3x}{12} - \frac{5}{4} - \frac{x}{8} + 1 = 0. \quad (1)$$

Simplifying, $\frac{5x}{8} = \frac{5}{4}$; (2)

whence, $x = 2.$

54. The equation may be expressed:

$$\frac{x}{6} - \frac{4}{6} - \frac{x}{30} - \frac{1}{10} + \frac{3x}{4} = x - \frac{x}{4} + \frac{5}{6}. \quad (1)$$

Simplifying, $\frac{4x}{30} = \frac{48}{30}$; (2)

whence, $x = 12.$

55. The equation may be written:

$$\frac{13}{15} - \frac{4x}{15} - 1 - \frac{4x}{3} + 6x - 8 = \frac{12x}{5} + \frac{58}{15}. \quad (1)$$

Simplifying, $6x - 4x = 12$; (2)

whence, $x = 6.$

56. Clearing, $y^2 + 2y + 1 - y^2 + 2y - 1 = 3y + 3.$

Transposing, etc., $y = 3.$

57. Clearing, $3x + 18 + 7x - 28 = 80.$ (1)

Transposing, etc., $10x = 90$; (2)

$x = 9.$

58. The equation may be expressed:

$$x + \frac{2}{3} - \frac{6x - 3}{x + 4} - \frac{3x}{5} = \frac{2x}{5} - \frac{101}{24}. \quad (1)$$

Simplifying, $\frac{6x - 3}{x + 4} = \frac{117}{24}.$ (2)

Clearing, $144x - 72 = 117x + 468$; (3)

whence, $x = 20.$

59. Clearing, $4x^2 - 20x + 2x^2 + 6x - 6x^2 + 12x + 90 = x + 3.$

Transposing, etc., $$-3x = -87; \tag{1}$$

$$x = 29. \tag{2}$$

60. Clearing, $$bc(x-1) + ac(x-1) + ab(x-1) = 0. \tag{1}$$

$$(bc + ac + ab)(x-1) = 0; \tag{2}$$

$$(x-1) = 0;$$

$$x = 1.$$

61. $$(a^2 - b^2)y = a^2 + 2ab + b^2. \tag{1}$$

Dividing by $(a+b)$, $$(a-b)y = a + b; \tag{2}$$

$$y = \frac{a+b}{a-b}.$$

62. $$(x-a)^2 - (x-b)^2 = (a-b)^2. \tag{1}$$

Factoring, $$(x - a + x - b)(x - a - x + b) = (a-b)(a-b); \tag{2}$$

or, $$(2x - a - b)(b - a) = (a-b)(a-b); \tag{3}$$

dividing by $(b-a)$, $$2x - a - b = b - a; \tag{4}$$

whence, $$2x = 2b;$$

$$x = b.$$

63. The equation may be written:

$$\frac{ab^2 - az^2}{cz} - \frac{ab^2}{cz} - cz = -\frac{2z}{c} - \frac{8a}{c}. \tag{1}$$

Simplifying, $$-\frac{az^2}{cz} - cz = -\frac{2z + 8a}{c}. \tag{2}$$

Clearing, $$-az - c^2z = -2z - 8a. \tag{3}$$

Transposing, etc., $$(a + c^2 - 2)z = 8a;\ z = \frac{8a}{a + c^2 - 2}. \tag{4}$$

64. $$\frac{1}{m+n} - \frac{2mn}{(m+n)^3} = \frac{x}{(m+n)^2}. \tag{1}$$

Clearing, $$m^2 + 2mn + n^2 - 2mn = (m+n)x. \tag{2}$$

Transposing, etc., $$x = \frac{m^2 + n^2}{m+n}. \tag{3}$$

Page 134.

65. The equation may be written:

$$\frac{ax}{b}-\frac{2}{3}-\frac{ax}{2b}+\frac{a}{2b}=\frac{ax}{b}-\frac{2}{3}. \qquad (1)$$

Simplifying, $\frac{ax}{2b}=\frac{a}{2b}$; (2)

Clearing, $2abx=2ab$; (3)

whence, $x=1.$

66. Clearing, $ax+bx-2a-ax+2b+bx=c+ad+bd.$ (1)

Transposing, etc., $2bx=c+ad+bd+2a-2b$; (2)

whence, $x=\frac{c+ad+bd+2a-2b}{2b}.$ (3)

67. Clearing, $4x-4ab-10b^2+3x-3b^2=7a^2+22ab-6x+6a^2.$ (1)

Transposing, etc., $13x=13a^2+26ab+13b^2$; (2)

whence, $x=a^2+2ab+b^2.$ (3)

SIMPLE EQUATIONS—PROBLEMS.

68. Let x represent the number;

then, $x+\frac{x}{2}-60=65-x.$ (1)

Clearing, etc., $x=50.$ (2)

69. Let x represent the less number;

then, $x+8$ will represent the greater;

and, $\frac{x+8}{x}=3.$ (1)

Clearing, etc., $x=4$; (2)

and, $x+8=12.$ (3)

70. Let x represent his property;

then, $\frac{x}{2}+\frac{x}{6}+\frac{x}{12}+600=x.$ (1)

Clearing, etc., $x=2400$ dollars. (2)

71. Let x represent one number;

then, $70-x$ will represent the other;

and, $\frac{x-1}{70-x}=2.$ (1)

Clearing, etc., $x=47$, and $70-x=23.$ (2)

73. Let x represent the number of gallons;

then, $$\frac{x}{5}+10=\frac{x}{3}. \qquad (1)$$

Clearing, etc., $$x=75 \text{ gallons}. \qquad (2)$$

Page 135.

74. Let x represent the number of days it will take both to do it;

then, $\frac{1}{x}=$ the part both can do in 1 day;

$\frac{1}{9}=$ the part A can do in 1 day;

$\frac{1}{10}=$ the part B can do in 1 day.

Therefore, $$\frac{1}{9}+\frac{1}{10}=\frac{1}{x}; \qquad (1)$$

whence, $$x=4\tfrac{14}{19} \text{ days}. \qquad (2)$$

75. Let x represent the number of days it will take all to do it;

then, $\frac{1}{x}=$ the part all can do in 1 day;

$\frac{1}{5}=$ the part A can do in 1 day;

$\frac{1}{7}=$ the part B can do in 1 day;

$\frac{1}{9}=$ the part C can do in 1 day.

Therefore, $$\frac{1}{5}+\frac{1}{7}+\frac{1}{9}=\frac{1}{x}; \qquad (1)$$

whence, $$x=2\tfrac{29}{143} \text{ days}. \qquad (2)$$

76. Let x represent the time in which both can fill it;

then, $\frac{1}{x}=$ the part both can fill in 1 hour;

$\frac{1}{8}=$ the part one can fill in 1 hour;

$\frac{1}{9}=$ the part the other can fill in 1 hour.

Therefore, $$\frac{1}{8}+\frac{1}{9}=\frac{1}{x}; \qquad (1)$$

whence, $$x=4\tfrac{4}{17} \text{ hours}. \qquad (2)$$

77. Let x represent the time required to fill it, when both are running;

then, $\frac{1}{x}=$ the part filled per hour when both are running;

$\frac{1}{3}=$ the part 1st pipe can fill per hour;

$\frac{1}{4}=$ the part 2d pipe can empty per hour.

Therefore, $$\frac{1}{3}-\frac{1}{4}=\frac{1}{x}; \qquad (1)$$

whence, $$x=12 \text{ hours}. \qquad (2)$$

78. Let x represent the weight of the body;

then, $\frac{x}{2} + 9 =$ weight of head;

and, $x = \frac{x}{2} + 18;$ (1)

whence, $x = 36.$ (2)

Therefore, $36 + 27 + 9 = 72$ pounds, the weight of fish. (3)

79. Let x represent the number in the detachment;

then, $\frac{2x}{3} =$ the number on duty;

$\frac{x}{8} =$ the number sick;

$\frac{x}{24} =$ the number absent on leave.

Therefore, $\frac{2x}{3} + \frac{x}{8} + \frac{x}{24} + 380 = x;$ (1)

whence, $x = 2280$ soldiers in detachment. (2)

Page 136.

81. Let $10x$ represent B's share;

then, $8x$ will represent A's share;

$18x$ will represent C's share;

$13x$ will represent D's share.

Therefore, $10x + 8x + 18x + 13x = 392$ A.; (1)

whence, $x = 8;$ (2)

and, $10x = 80$ A.; $8x = 64$ A.; $18x = 144$ A.; $13x = 104$ A.

82. Let $6x$ represent the quantity of corn;

then, $4x =$ the quantity of oats;

$2x =$ the quantity of rye.

Therefore, $6x + 4x + 2x = 300;$ (1)

whence, $x = 25;$ (2)

and, $6x = 150$ bu.; $4x = 100$ bu.; $2x = 50$ bu.

83. Let x represent the greater;

then, $204 - x =$ the less.

Therefore, $204 - x - \frac{2x}{5} = x - \frac{3}{7}(204 - x);$ (1)

or, $204 - \frac{7x}{5} = x - \frac{612}{7} + \frac{3x}{7}.$ (2)

Clearing, etc., $x = 103\frac{1}{33};$ (3)

$204 - x = 100\frac{32}{33}.$ (4)

84. Let x represent his money;

then, $14 + \frac{3x}{7} + 6 + \frac{x}{3} = x.$

Clearing, etc., $x = 84$ dollars.

85. Let x represent his capital;

then, $\frac{4x}{5} =$ the capital at end of first year;

$\frac{12x}{40} =$ the gain during second year;

$\frac{44x}{40} =$ the capital at end of second year;

$\frac{12x}{40} =$ the gain during third year;

$\frac{56x}{40} =$ the capital at end of third year.

Therefore, $\frac{56x}{40} = 7000;$ (1)

whence, $x = 5000$ dollars. (2)

Page 137.

87. Let $x =$ number on a side;

then, $x^2 + 295 =$ entire number of men;

$(x + 5)^2 = x^2 + 295;$ (1)

whence, $10x = 270;$ (2)

$x = 27$, number of men on a side;

$x^2 + 295 = 1024$, entire number of men.

88. Let x represent the number of men on each side in first arrangement;

then, $x^2 + 31 =$ the entire number;

$x + 1 =$ the number of men on each side in second arrangement;

then, $(x + 1)^2 - 24 =$ the entire number.

Therefore, $x^2 + 31 = x^2 + 2x - 23;$ (1)

whence, $x = 27.$ (2)

Therefore, $(27 \times 27) + 31 = 760$, the entire number.

89. Let x represent the number of pounds;

then, $11x - 30 = 10\frac{1}{2}x + 15;$ (1)

whence, $x = 90.$ (2)

91. Let $5x$ represent one number;

then, $7x =$ the other;

and, $5x + 7x = 72;$ (1)

whence, $x = 6.$ (2)

Therefore, $5x = 30;\ 7x = 42.$

92. Let $3x$ represent A's age;

then, $8x =$ B's age;

and, $3x + 8x = 44;$ (1)

whence, $x = 4.$ (2)

Therefore, $3x = 12;\ 8x = 32.$

Page 138.

93. Let $8x$ represent the share of elder;

then, $7x =$ the share of younger;

and, $8x + 7x = 15000;$ (1)

whence, $x = 1000.$ (2)

Therefore, $8x = 8000$ dollars, elder's share;

$7x = 7000$ dollars, younger's share.

94. Let $5x$ represent A's share;

then, $3x =$ B's share;

and, $8x =$ whole sum.

Therefore, $5x - \frac{40x}{9} = 50;$ (1)

whence, $x = 90;$ (2)

and, $5x = 450$ dollars, A's share;

$3x = 270$ dollars, B's share.

95. Let x represent the amount each had at first;

then, $x + 20 =$ amount A had after debt was paid;

$x - 20 =$ amount B had after paying A;

$\frac{x}{2} + 10 =$ amount A paid B afterward;

$\frac{x}{2} + 10 =$ amount A had after paying B;

$\frac{3x}{2} - 10 =$ amount B had after A paid him.

Therefore, $\frac{x}{2} + 10 = \frac{3x}{4} - 5;$ (1)

whence, $x = 60$ dollars.

96. Let x represent the number of men;

then, $2x - 2 =$ the number of women;

and, $6x - 10 =$ the number of children;

$12x =$ sum paid men;

$12x - 12 =$ sum paid women;

$18x - 30 =$ sum paid children.

Therefore,

$12x + 12x - 12 + 18x - 30 = 252;$ (1)

whence, $x = 7;$ (2)

hence, $2x - 2 = 12;\ 6x - 10 = 32;$

and, $7 + 12 + 32 = 51$, the number of persons.

97. Let x represent the number of apples;

then, $\frac{2x}{5} =$ cost of apples;

$\frac{x}{4} =$ selling price of one half of the apples;

$\frac{x}{6} =$ selling price of other half of the apples.

Therefore, $\frac{x}{4} + \frac{x}{6} = \frac{2x}{5} + 1;$ (1)

whence, $x = 60$ apples bought. (2)

98. Let x represent the original capital;

then, $x - 1000 =$ the gain of first year;

$2x - 2000 =$ the gain of second year;

$4x - 4000 =$ the gain of third year;

hence, $x + x - 1000 + 2x - 2000 + 4x - 4000 = 3x;$ (1)

whence, $x = 1400$ dollars. (2)

99. Let x represent the capital of each;

then, $\frac{4x}{3} =$ A's capital at end of first year;

$\frac{3x}{4} =$ B's capital at end of first year;

$\frac{4x}{3} - 72 =$ A's capital at end of second year;

$\frac{3x}{4} + 36 =$ B's capital at end of second year.

Therefore, $\frac{3}{4}\left(\frac{4x}{3} - 72\right) = \frac{3x}{4} + 36$; (1)

whence, $x = 360$ dollars. (2)

Page 139.

100. Let x re[illegible]esent the quantity conveyed by first per minute;

then, $x + 6 =$ the quantity conveyed by second per minute;

and, $2x + 6 =$ the quantity conveyed by both per minute.

Therefore, $\frac{648}{2x + 6} = 18$; (1)

whence, $x = 15$; and $x + 6 = 21$ gallons. (2)

101. Let x represent the number in first field;

then, $x + 4 =$ the number in second field;

$\frac{x}{2} + 1 =$ the number in third field;

$2x + 4 =$ the number in fourth field.

Therefore, $x + x + 4 + \frac{x}{2} + 1 + 2x + 4 = 90$; (1)

whence, $x = 18$; (2)

hence, the numbers are 18, 22, 10, 40.

102. Let x represent amount paid for house;

then, $10000 - x =$ the amount invested;

$\frac{6}{100}$ of $\frac{1}{3}(10000 - x)$, or $\frac{1}{50}(10000 - x) =$ interest on first investment;

$\frac{5}{100}$ of $\frac{2}{3}(10000 - x)$, or $\frac{1}{30}(10000 - x) =$ interest on second investment.

Therefore,

$\frac{1}{50}(10000 - x) + \frac{1}{30}(10000 - x) = 320.$ (1)

Uniting, $\frac{8}{150}(10000 - x) = 320.$ (2)

Dividing by $\frac{3}{150}$, $10000 - x = 6000$; (3)
whence, $x = 4000$ dollars. (4)

103. Let x represent the side of the square;
then, $x^2 =$ the number of square yards;
and, $4x =$ the perimeter.
Therefore, $40x^2 = 400x$; (1)
whence, $x = 10$ yards. (2)

104. Let x represent the rate of second per day;
then, $9x =$ the distance traveled back by first;
$\frac{9x}{12} =$ the time required for first to travel back;
$108 - 9x =$ the distance of first from starting-point when he turns and pursues his journey;
$18x =$ distance from starting-point to fort
$18x - (108 - 9x) =$ distance first must travel from place where he turns and pursues his journey to the f[illegible];
$\frac{18x - (108 - 9x)}{12} =$ time required by first to travel the d[illegible]ance.
Therefore,

$$9 + \frac{9x}{12} + \frac{18x - (108 - 9x)}{12} = 18; \quad (1)$$

whence, $x = 6$ miles per day. (2)

105. Let x represent the number bought;
then, $\frac{5x}{4} =$ the cost;
and, $\frac{4x}{3} =$ the selling price.
Therefore, $\frac{4x}{3} - \frac{5x}{4} = 60$; (1)
whence, $x = 720$ apples bought. (2)

106. Let $24x$ represent A's share;
then, $36x =$ B's share;
$80x =$ C's share;
$175x =$ D's share.
Therefore, $24x + 36x + 80x + 175x = 315$; (1)
whence, $x = 1$. (2)
The shares are: A's, \$24; B's, \$36; C's, \$80; D's, \$175.

107. Let $2x$ and $3x$ represent the numbers;

then, $\frac{1}{2}(3x - 50) = 2x - 50;$ (1)

whence, $x = 50.$ (2)

And the numbers are 100 and 150.

Page 140.

108. Let x represent the price of apples per dozen;

then, $x + 5 =$ the price of eggs per dozen.

Therefore, $8x + 40 = 13\frac{5}{7}x;$ (1)

whence, $x = 7$ cents; (2)

and, $x + 5 = 12$ cents.

109. Let x represent the number of days A works;

then, $2x =$ the number of days B works;

$\frac{3x}{2} =$ the number of days C works;

$7x =$ the number of rods A builds;

$12x =$ the number of rods B builds;

$\frac{15x}{2} =$ the number of rods C builds.

Therefore, $7x + 12x + \frac{15x}{2} = 318;$ (1)

whence, $x = 12.$ (2)

And A works 12 days, B 24 days, C 18 days.

110. Let x represent the value of poorer horse;

then, $\frac{x + 150}{2} =$ the value of better horse.

Therefore, $\frac{x + 150}{2} + 150 = 3x.$ (1)

Clearing, etc., $x = 90$ dollars, value of poorer horse; (2)

and, $\frac{x + 150}{2} = 120$ dollars, value of better horse. (3)

111. Let x represent the length of shorter piece;

then, $4x - 12 =$ the length of longer piece;

$5x - 25 =$ the selling price of remnant of shorter;

$24x - 210 =$ the selling price of remnant of longer.

Therefore, $5x - 25 + 24x - 210 = 142;$ (1)

whence, $x = 13$ yards, length of shorter piece, (2)

and $4x - 12 = 40$ yards, length of longer piece. (3)

113. Let x represent the number of minute-spaces the minute-hand travels before the hands are together;

then, $\frac{x}{12}$ = the number of minute-spaces the hour-hand travels in the same time.

Since the hands are 25 minute-spaces apart at 5 o'clock, when they come together the minute-hand must have traveled 25 minute-spaces farther than the hour-hand.

Therefore, $$x - \frac{x}{12} = 25;$$

whence, $x = 27\frac{3}{11}$, the number of minutes after 5 o'clock.

Page 141.

114. For explanation of statement, see example 113.

$$x - \frac{x}{12} = 40;$$

whence, $x = 43\frac{7}{11}$, the number of minutes after 8 o'clock.

115. For explanation for part of statement, see example 113.

It is evident that, in order to make a straight line after 4 o'clock, the minute-hand must travel 50 minute-spaces farther than the hour-hand.

Therefore, $$x - \frac{x}{12} = 50;$$

whence, $x = 54\frac{6}{11}$, the number of minutes after 4 o'clock.

116. For explanation of part of statement, see example 113.

It is evident that, in order to make a straight line after 5 o'clock, the minute-hand must travel 55 minute-spaces farther than the hour-hand.

Therefore, $$x - \frac{x}{12} = 55;$$

whence, $x = 60$, the number of minutes after 5 o'clock, or 6 o'clock.

117. For explanation of statement, see example 113.

$$x - \frac{x}{12} = 15;$$

whence, $x = 16\frac{4}{11}$, the number of minutes after 6 o'clock.

118. For explanation of part of statement, see example 113.

It is evident that, in order to have the hour and minute-hands 15 minute-spaces apart after half-past 8 o'clock, the minute-hand must travel $27\frac{1}{2}$ minute-spaces farther than the hour-hand after half-past 8 o'clock.

Therefore, $x - \frac{x}{12} = 27\tfrac{1}{2}$;

whence, $x = 30$, the number of minutes after half-past 8 o'clock, or 9 o'clock.

119. Let $x =$ the value of the better horse;

$460 - x =$ the value of the poorer horse;

then, $x + 210 = 3(460 - x) + \frac{x}{10}$; (1)

$10x + 2100 = 13800 - 30x + x$; (2)

whence, $x = 300$ dollars, value of better horse.

$460 - x = 160$ dollars, value of poorer horse.

120. Let $x =$ daily wages of laborer;

$2x =$ daily wages of first mechanic;

$3x =$ daily wages of second mechanic;

then, $10x + 14x + 12x = 72$; (1)

$36x = 72$; (2)

$x = 2$, daily wages of laborer,

$2x = 4$, daily wages of first mechanic;

$3x = 6$, daily wages of second mechanic.

121. Let $x =$ number of miles;

then, $\frac{x}{8} =$ time to go;

and, $\frac{x}{3\frac{1}{2}} =$ time to return;

then, $\frac{x}{8} + \frac{2x}{7} = 11\tfrac{1}{2}$; (1)

$7x + 16x = 644$; (2)

whence, $x = 28$ miles.

122. Let $x =$ one;

and, $80 - x =$ the other; (1)

then, $x - (80 - x) = 6$; (2)

$2x = 86$;

$x = 43$, first;

$80 - x = 37$, second.

123. Let $x =$ what each had at first;
then, $x + 60 - 10 =$ James' after the exchange;
and, $x - 60 + 10 =$ Isaac's after the exchange;
$3(x - 60 + 10) = x + 60 - 10;$
whence, $x = 100$ dollars.

124. Let $x =$ the first;
$\frac{2}{3}(x - 20) =$ the second;
$2x - 50 =$ the third;
then, $x + \frac{2x - 40}{3} + 2x - 50 = 230.$ (1)
Clearing, $3x + 2x - 40 + 6x - 150 = 690.$ (2)
Transposing, etc., $11x = 880;$
$x = 80$, first;
$\frac{2}{3}(x - 20) = 40$, second;
$2x - 50 = 110$, third.

Page 142.

125. Let $x =$ number of days he worked;
then, $3\frac{1}{2}x - 24 = 39;$ (1)
whence, $7x = 126;$ (2)
$x = 18$, days worked.

126. Let $x =$ rods wide;
$2x =$ rods long;
$(2x + 1)(x - 1) = 6;$ (1)
$2x^2 - x - 1 = 2x^2 - 6;$ (2)
$x = 5$ rods, width;
$2x = 10$ rods, length.

127. Let $x =$ the distance;
then the second man traveled x miles in the same time that the first one traveled $x - 30$.

Therefore, $\frac{x - 30}{5} = \frac{x}{8};$ (1)
$8x - 240 = 5x;$ (2)
whence, $x = 80$ miles, the distance.

128. Let $x =$ number of dimes;

$4x =$ number of dollars;

then, $$4x + \frac{x}{10} = 28.70; \tag{1}$$

$$40x + x = 287; \tag{2}$$

whence, $41x = 287$;

$x = 7$, number of dimes;

$4x = 28$, number of dollars.

129. Let $x =$ sum at first;

$\frac{4x}{5} + 4000 =$ sum at beginning of second year;

$\frac{16x}{25} + 7200 =$ sum at beginning of third year;

$\frac{64x}{125} + 9760 =$ sum at beginning of fourth year;

$$\frac{64x}{125} + 9760 = 13600;$$

whence, $x = 7500$ dollars, sum in bank at first.

130. Let $x =$ hours to overtake the second footman;

then, $10x = 35 + 5x$;

whence, $x = 7$ hours.

Let x again $=$ hours to overtake first footman;

then, $10x = 45\frac{1}{2} + 3\frac{1}{2}x$;

whence, $x = 7$.

Therefore, in 7 hours the horseman overtook the two footmen, who had reached the same point in the journey.

131. Let $x =$ number of cents;

$\frac{x}{2} - \frac{1}{2} =$ left 1st time;

$\frac{x}{4} - \frac{3}{4} =$ left 2d time;

$\frac{x}{8} - \frac{7}{8} =$ left 3d time;

then, $$\frac{x}{8} - \frac{7}{8} = 2;$$

whence, $x = 23$ cents at first.

Page 143.

133. Let x represent the value of the saddle;

then, $nx =$ the value of the horse.

Therefore, $x + nx = m;$ (1)

whence, $x = \frac{m}{1+n}$, value of saddle; (2)

and, $nx = \frac{mn}{1+n}$, value of horse. (3)

Giving the letters their values,

$$\frac{m}{1+n} = \frac{200}{1+9} = 20 \text{ dollars, value of the saddle;}$$

and, $$\frac{mn}{1+n} = \frac{200 \times 9}{1+9} = 180 \text{ dollars, value of the horse.}$$

134. Let x represent what he gave B;

then, $ax =$ what he gave A.

Therefore, $x + ax = b;$ (1)

whence, $x = \frac{b}{1+a}$, what he gave B; (2)

and, $ax = \frac{ab}{1+a}$, what he gave A. (3)

Giving the letters their values,

$$\frac{b}{1+a} = \frac{75}{1+4} = 15 \text{ dollars, what he gave B;}$$

and, $$\frac{ab}{1+a} = \frac{4 \times 75}{1+4} = 60 \text{ dollars, what he gave A.}$$

135. Let x represent one part;

then, $ax =$ the other part.

Therefore, $x + ax = b;$ (1)

whence, $x = \frac{b}{1+a}$, one part; (2)

and, $ax = \frac{ab}{1+a}$, the other part. (3)

Giving the letters their values,

$$\frac{b}{1+a} = \frac{24}{1+7} = 3, \text{ one part;}$$

and, $$\frac{7 \times 24}{1+7} = 21, \text{ the other part.}$$

136. Let x represent the number of days in which both can do it;

then, $\frac{1}{x} =$ the part both can do in one day;

$\frac{1}{n} =$ the part A can do in one day;

$\frac{1}{m} =$ the part B can do in one day.

Therefore, $$\frac{1}{n} + \frac{1}{m} = \frac{1}{x}; \qquad (1)$$

whence, $x = \frac{mn}{m+n}$, the number of days it takes both to do it. (2)

Giving the letters their values,

$$\frac{mn}{m+n} = \frac{7 \times 5}{7+5} = 2\tfrac{11}{12} \text{ days}; \qquad (3)$$

and, $$\frac{mn}{m+n} = \frac{8 \times 10}{8+10} = 4\tfrac{4}{9} \text{ days}. \qquad (4)$$

137. Let x represent the sum paid by each;

then, $ax =$ the entire expense.

Therefore, $$\frac{ax}{a+b} = x - d; \qquad (1)$$

whence, $x = \frac{ad + bd}{b}$, the sum each paid. (2)

Giving the letters their values,

$$\frac{ad + bd}{b} = \frac{(8 \times 1) + (4 \times 1)}{4} = 3 \text{ dollars, each paid}. \qquad (3)$$

138. Let x represent the number;

then, $$\frac{x}{b} + x + b = c; \qquad (1)$$

whence, $x = \frac{bc - b^2}{1 + b}$, the number. (2)

Giving the letters their values,

$$\frac{bc - b^2}{1 + b} = \frac{(16 \times 84) - 16^2}{1 + 16} = 64. \qquad (3)$$

SIMULTANEOUS EQUATIONS.

Page 151.

16. Eq. (1) × 4, $$\frac{4}{x} + \frac{8}{y} = 40. \qquad (3)$$

(3) − (1), $$\frac{5}{y} = 20. \qquad (4)$$

$$y = \tfrac{1}{4}. \qquad (5)$$

Substituting (5) in (1), $$x = \tfrac{1}{2}. \qquad (6)$$

17. (1) × 3, $\frac{21}{x} + \frac{15}{y} = 57.$ (3)

(2) × 4, $\frac{40}{x} - \frac{15}{y} = 35.$ (4)

(3) + (4), $\frac{61}{x} = 92.$ (5)

$x = \frac{61}{92}.$ (6)

Substituting (6) in (1), $y = \frac{61}{103}.$ (7)

18. (2) × 4, $\frac{14}{3x} - \frac{2}{5y} = 12.$ (3)

(1) + (3), $\frac{19}{3x} = 19;$ (4)

$x = \frac{1}{3}.$ (5)

Substituting (5) in (1), $y = \frac{1}{5}.$ (6)

19. (1) + (2), $\frac{2a}{x} = m + n;$ (3)

$x = \frac{2a}{m+n}.$ (4)

(1) − (2), $\frac{2b}{y} = m - n;$ (5)

$y = \frac{2b}{m-n}.$ (6)

20. Transposing and clearing (1), $x - 2y = -80.$ (3)

Clearing (2), $12x + 12y + 20y = 30x - 15y + 2100.$ (4)

Combining, etc. (4), $-18x + 47y = 2100.$ (5)

(3) × 18, $18x - 36y = -1440.$ (6)

(5) + (6), $11y = 660;$ (7)

$y = 60.$ (8)

Substituting (8) in (3), $x = 40.$

21. Clearing (1), $2x - y + 21x = 14y - 42.$ (3)

Combining, etc., $23x - 15y = -42.$ (4)

Clearing (2), $6y + 18 + 5y - 5x = 60x - 240.$ (5)

Combining, etc., $-65x + 11y = -258.$ (6)

(4) × 11, $253x - 165y = -462.$ (7)

(6) × 15, $-975x + 165y = -3870.$ (8)

(7) + (8), $-722x = -4332;$ (9)

$x = 6.$ (10)

Substituting (10) in (4), $y = 12.$

22. Clearing (1), $12x - 24 - 200 + 20x = 15y - 150.$ (3)

Combining, etc., $32x - 15y = 74.$ (4)

Clearing (2), $16y + 32 = 12x + 3y + 39.$ (5)

Combining, etc., $12x - 13y = -7.$ (6)

(4) × 3, $96x - 45y = 222.$ (7)

(6) × 8, $96x - 104y = -56.$ (8)

(7) – (8), $59y = 278;$ (9)

$y = 4\frac{42}{59}.$ (10)

Substituting (10) in (6), $x = 4\frac{123}{236}.$ (11)

23. From (2), $x - 1 = y.$ (3)

Substituting this value of $x - 1$ in second fraction in (1), we have:

$$\frac{x+1}{y-1} - 1 = \frac{6}{7}. \qquad (4)$$

Transposing and clearing (4), $7x + 7 = 13y - 13.$ (5)

Transposing, etc., (5), $7x - 13y = -20.$ (6)

(2) × 7, $7x - 7y = 7.$ (7)

(6) – (7), $6y = 27;$ (8)

$y = 4\frac{1}{2}.$ (9)

Substituting (9) in (2), $x = 5\frac{1}{2}.$ (10)

24. Clearing (1), $5 - 15x + 21y - 7 = 70.$ (3)

Combining, $-15x + 21y = 72.$ (4)

Clearing (2), $3x + y + 11y = 99.$ (5)

Combining, $3x + 12y = 99.$ (6)

(6) × 5, $15x + 60y = 495.$ (7)

(4) + (7), $81y = 567.$ (8)

$y = 7.$ (9)

Substituting (9) in (6), $x = 5.$ (10)

25. Clearing (2), $3y = 35x - 5y - 23x.$ (3)

Combining, $8y = 12x;$ (4)

$y = \frac{3}{2}x.$ (5)

Substituting (5) in (1), $x = 2.$ (6)

Substituting (6) in (5), $y = 3.$ (7)

Page 152.

26. Clearing (1), $bx + ay = 2ab.$ (3)

(3) + (2), $2bx = 2ab;$ (4)

$x = a.$ (5)

(3) − (2), $2ay = 2ab;$ (6)

$y = b.$ (7)

27. Clearing (1), $x + y = c.$ (3)

Clearing (2), $ax - by = ac - bc$ (4)

(3) × a, $ax + ay = ac.$ (5)

(5) − (4), $(a + b)y = bc;$ (6)

$y = \dfrac{bc}{a+b}.$ (7)

(3) × b, $bx + by = bc.$ (8)

(4) + (8), $(a + b)x = ac;$ (9)

$x = \dfrac{ac}{a+b}.$ (10)

28. Clearing (1), $3a^2 + ax = b^2 + by.$ (3)

Transposing, $ax - by = b^2 - 3a^2.$ (4)

(2) − (4), $3by = d - b^2 + 3a^2;$ (5)

$y = \dfrac{d - b^2 + 3a^2}{3b}.$ (6)

(4) × 2, $2ax - 2by = 2b^2 - 6a^2.$ (7)

(7) + (2), $3ax = d + 2b^2 - 6a^2;$ (8)

$x = \dfrac{d + 2b^2 - 6a^2}{3a}.$ (9)

29. Multiplying (1) by n, $\dfrac{an}{x} + \dfrac{bn}{y} = cn;$ (3)

and (2) by b, $\dfrac{bm}{x} + \dfrac{bn}{y} = be.$ (4)

Subtracting (4) from (3), $\dfrac{an - bm}{x} = cn - be;$ (5)

whence, $(cn - be)x = an - bm;$ (6)

and, $x = \dfrac{an - bm}{cn - be}.$

Multiplying (1) by m, $\frac{am}{x}+\frac{bm}{y}=cm$; (7)

and (2) by a, $\frac{am}{x}+\frac{an}{y}=ae.$ (8)

Subtracting (7) from (8), whence, $y=\frac{an-bm}{ae-cm}.$

30. Subtracting (2) from (1), $(b-c)y=r-s$; (3)

whence, $y=\frac{r-s}{b-c}.$

Multiplying (1) by c, $acx+bcy=cr$; (4)

and (2) by b, $abx+bcy=bs.$ (5)

Subtracting (4) from (5), $(ab-ac)x=bs-cr$; (6)

whence, $x=\frac{bs-cr}{a(b-c)}.$

31. (1) + (2), $\frac{2x}{m}=3$; (3)

$x=\frac{3m}{2}.$ (4)

(1) − (2), $\frac{2y}{n}=1$; (5)

$y=\frac{n}{2}.$ (6)

32. Clearing (2), $abdx-bcdy=d-b.$ (3)

$(1)\times d$, $abdx+cd^2y=2d.$ (4)

(4) − (3), $cd^2y+bcdy=d+b$; (5)

$$y=\frac{d+b}{cd^2+bcd}=\frac{d+b}{cd(d+b)}=\frac{1}{cd}. \quad (6)$$

$(1)\times b$, $ab^2x+bcdy=2b.$ (7)

(3) + (7), $abdx+ab^2x=d+b$; (8)

$$x=\frac{d+b}{abd+ab^2}=\frac{d+b}{ab(d+b)}=\frac{1}{ab}. \quad (9)$$

33. Multiplying (1) by n, $anx-dny=cn$; (3)

and (2) by d, $dmx-dny=cd.$ (4)

Subtracting (4) from (3), $(an-dm)x=c(n-d)$; (5)

whence, $x=\frac{c(n-d)}{an-dm}.$

Substituting in (1), $y=\frac{c(m-a)}{an-dm}.$

34. Clearing, $bx + ay = 2a^2b^2$; (1)

$x + y = a^2b + ab^2$. (2)

Multiplying (2) by a, $ax + ay = a^3b + a^2b^2$. (3)

Subtracting (1) from (3), $(a - b)x = a^2b(a - b)$; (4)

whence, $x = a^2b$.

Substituting in (2), $y = ab^2$.

Page 153.

2. Multiplying (2) by 2, $\frac{12}{x} - \frac{4}{y} = 28$. (3)

Add (3) and (1), $\frac{14}{x} = 42$; (4)

$\frac{1}{x} = 3$; $x = \frac{1}{3}$.

Substituting in (1), $y = \frac{1}{2}$.

3. Multiplying (1) by 5, $\frac{5}{x} + \frac{15}{y} = 55$. (3)

Subtracting (2) from (3), $\frac{11}{y} = 33$; (4)

$\frac{1}{y} = 3$; $y = \frac{1}{3}$.

Substituting in (1), $x = \frac{1}{2}$.

4. Multiplying (1) by 5, $\frac{10}{x} + \frac{5}{y} = \frac{20}{3}$. (3)

Subtracting (2) from (3), $\frac{7}{x} = \frac{21}{6}$; (4)

$\frac{1}{x} = \frac{3}{6}$; $x = 2$.

Substituting in (1), $y = 3$.

5. Multiplying (1) by 5, $\frac{5}{x} + \frac{15}{y} = 5a$. (3)

Subtracting (2) from (3), $\frac{13}{y} = 5a - b$; (4)

$y = \frac{13}{5a - b}$.

Substituting in (1), $x = \frac{13}{3b - 2a}$.

6. Multiplying (1) by 5, $\frac{40}{x}+\frac{30}{y}=15.$ (3)

Multiplying (2) by 2, $\frac{12}{x}+\frac{30}{y}=8.$ (4)

Subtracting (4) from 3, $\frac{28}{x}=7;\ 7x=28;\ x=4.$

Substituting in (1), $y=6.$

7. Multiplying (1) by 6, $\frac{30}{x}+\frac{36}{y}=42.$ (3)

Multiplying (2) by 4, $\frac{28}{x}+\frac{36}{y}=40.$ (4)

Subtracting (4) from (3), $\frac{2}{x}=2;\ \frac{1}{x}=1;\ x=1.$

Substituting in (1), $y=3.$

8. Multiplying (2) by 3, $\frac{9}{x}+\frac{30}{y}=\frac{87}{6}.$ (3)

Subtracting (1) from (3), $\frac{22}{y}=\frac{44}{6};$ (4)

Dividing by 22, $\frac{1}{y}=\frac{2}{6};\ y=3.$

Substituting in (1), $x=2.$

9. Multiplying (1) by a, $\frac{a^2}{x}+\frac{ab}{y}=ac.$ (3)

Multiplying (2) by b, $\frac{b^2}{x}+\frac{ab}{y}=bd.$ (4)

Subtracting (4) from (3), $\frac{a^2-b^2}{x}=ac-bd;$ (5)

whence, $x=\frac{a^2-b^2}{ac-bd}.$

Substituting in (1), $y=\frac{a^2-b^2}{ad-bc}.$

10. Multiplying (1) by 7, $\frac{42}{x}+\frac{56}{y}=7;$ (3)

Multiplying (2) by 6, $\frac{42}{x}-\frac{66}{y}=-54.$ (4)

Subtracting (4) from (3), $\frac{122}{y}=61;$ (5)

Dividing by 122, $\frac{1}{y}=\frac{1}{2};\ y=2.$

Substituting in (1), $x=-2.$

11. (1) may be written, $\frac{2}{x}+\frac{2}{y}=14.$ (1)

(2) may be written, $\frac{1}{x}+\frac{2}{y}=11.$ (2)

Subtracting (2) from (1), $\frac{1}{x}=3;\ x=\frac{1}{3}.$

Substituting in (1), $y=\frac{1}{4}.$

12. Multiplying (1) by 2, $\frac{6}{x+3}+\frac{12}{y-2}=14.$ (3)

Multiplying (2) by 3, $\frac{15}{x+3}+\frac{12}{y-2}=17.$ (4)

Subtracting (3) from (4), $\frac{9}{x+3}=3;$ (5)

$x+3=3;\ x=0.$

Substituting in (1), $y=3.$

13. Clearing, $bx+ay=abp;$ (1)

$ax+by=abq.$ (2)

Multiplying (1) by b, $b^2x+aby=ab^2p.$ (3)

Multiplying (2) by a, $a^2x+aby=a^2bq.$ (4)

Subtracting (3) from (4), $(a^2-b^2)x=ab(aq-bp);\ x=\frac{ab(aq-bp)}{a^2-b^2}.$

Substituting in (1), $y=\frac{ab(ap-bq)}{a^2-b^2}.$

SIMULTANEOUS EQUATIONS—PROBLEMS.

Page 155.

4. Let x and y represent the numbers;

then, $x+y=24;$ (1)

and, $x-y=8;$ (2)

whence, $x=16;\ y=8.$

5. Let x and y represent the numbers;

then, $x+y=29;$ (1)

and, $x-y=5;$ (2)

whence, $x=17;\ y=12.$

6. Let x and y represent the numbers;

then, $\frac{x+y}{2}=24;$ (1)

and, $\frac{x-y}{2}=17;$ (2)

whence, $x=41;\ y=7.$

7. Let x represent the daily wages of men, and y the wages of boys;

then, $6x + 2y = 28;$ (1)

and, $3x + 4y = 20;$ (2)

whence, $x = 4$ dollars; $y = 2$ dollars.

8. Let x represent the numerator, and y the denominator;

then, $\frac{x}{y} =$ the fraction;

and, $\frac{x+3}{y} = \frac{1}{3};$ (1)

$\frac{x}{y-1} = \frac{1}{5};$ (2)

whence, $x = 4;\ y = 21;\ \frac{x}{y} = \frac{4}{21}.$

9. Let x represent the value of first horse, and y the value of second horse;

then, $x + 10 = 2y;$ (1)

and, $y + 10 = x - 13;$ (2)

whence, $x = 56$ dollars; $y = 33$ dollars.

10. Let x represent the sum in the first, and y the sum in the second;

then, $x + y = 300;$ (1)

and, $x - 30 = y + 30;$ (2)

whence, $x = 180$ dollars; $y = 120$ dollars.

11. Let x represent A's money, and y B's money;

then, $x + y = 570;$ (1)

and, $3x + 5y = 2350;$ (2)

whence, $x = 250$ dollars; $y = 320$ dollars.

12. Let x represent the numerator, and y the denominator;

then, $\frac{x}{y} =$ the fraction;

and, $\frac{x+4}{y} = \frac{1}{2}.$ (1)

$\frac{x}{y+7} = \frac{1}{5};$ (2)

whence, $x = 5;\ y = 18;\ \frac{x}{y} = \frac{5}{18}.$

13. Let x and y represent the digits;

then, $10x + y =$ the number;

and, $10x + y = 4x + 4y;$ (1)

$10x + y + 18 = 10y + x;$ (2)

whence, $x = 2;\ y = 4;$ and the number is 24.

14. Let x represent the number of one kind, and y the number of the other;

then, $x + y = 6;$ (1)

and, $\frac{x}{10} + \frac{y}{2} = 1;$ (2)

whence, $x = 5;\ y = 1.$

Page 156.

15. Let x represent the number of persons, and y the amount paid by each;

then, $\frac{xy}{x+3} = y - 1;$ (1)

and, $\frac{xy}{x-2} = y + 1;$ (2)

whence, $x = 12$ persons; $y = 5$ dollars.

16. Let x represent the price of port, and y the price of sherry;

then, $20x + 30y = 120;$ (1)

and, $30x + 25y = 140;$ (2)

whence, $x = £3;\ y = £2.$

17. Let x and y represent the digits;

then, $10x + y =$ the number;

and, $x + y + 7 = 3x;$ (1)

$10x + y - 18 = 10y + x;$ (2)

whence, $x = 5;\ y = 3;$ and the number is 53.

18. Let x represent A's capital, and y B's;

then, $x + y = 9800;$ (1)

and, $\frac{5x}{6} = \frac{4y}{5};$ (2)

whence, $x = 4800$ dollars; $y = 5000$ dollars.

19. Let x represent the number of acres for which he paid \$20 per acre, and y the number of acres for which he paid \$30 per acre;

then, $x + y = 100;$ (1)

and, $20x + 30y = 2450;$ (2)

whence, $x = 55$ acres; $y = 45$ acres.

20. Let x represent the father's age, and y the son's;

then, $x + y = 80;$ (1)

and, $2y - 10 = x;$ (2)

whence, $x = 50$ years; $y = 30$ years.

21. Let x represent A's money, and y B's;

then, $\frac{x + 20}{4} = y - 20;$ (1)

and, $\frac{2}{3}(y + 20) = x - 20;$ (2)

whence, $x = 60$ cents; $y = 40$ cents.

22. Let x represent the number of acres at \$37, and y the number at \$45;

then, $x + y = 100;$ (1)

and, $37x + 45y = 4220;$ (2)

whence, $x = 65$ acres; $y = 35$ acres.

23. Let x represent the number of apples, and y the number of pears;

then, $\frac{x}{4} + \frac{y}{5} = 30;$ (1)

and, $\frac{x}{8} + \frac{y}{15} = 13;$ (2)

whence, $x = 72$ apples; $y = 60$ pears.

Page 157.

24. Let $x =$ cost of corn per bushel;

$y =$ cost of wheat per bushel;

then, $47x + 18y = 45.26;$ (1)

and, $3x + 63y = 64.74;$ (2)

whence, $x = 58¢;$ $y = \$1.00.$

25. Let $x =$ one;

$y =$ the other;

then, $\frac{x}{2} + \frac{2y}{3} = 38;$ (1)

and, $x + 3 = \frac{3}{8}(y - 8);$ (2)

whence, $x = 12;$ $y = 48.$

26. Let x = first boy's money;

y = second boy's money;

then, $x + \cancel{\frac{y}{2}} + .50 = 4\left(\cancel{\frac{y}{2}} - .50\right)$; (1)

and, $x - .50 = y + .50 - 2.50$; (2)

whence, $x = \$.50$; $y = \$2.00$.

27. Let x = number in 1st field;

y = number in 2d field;

then, $\frac{x}{4} = \frac{y}{6}$; (1)

and, $x - 10 = \frac{y + 10}{4}$; (2)

whence, $x = 20$ sheep;

and, $y = 30$ sheep.

28. Let x = Paul's age now;

y = Mabel's age now;

then, $x - 3 = 3\,(y - 3)$; (1)

and, $3\,(x + 3) = 5\,(y + 3)$; (2)

whence, $x = 12$ years;

$y = 6$ years.

29. Let $\frac{3}{x}$ = smaller fraction;

$\frac{3}{y}$ = larger fraction;

then, $\frac{3}{x} + \frac{3}{y} = \frac{9}{x}$; (1)

and, $\frac{6}{y} - \frac{9}{x} = \frac{3}{8}$; (2)

whence, $x = 8$;

$y = 4$;

and, $\frac{3}{8}$ = smaller;

$\frac{3}{4}$ = larger.

30. Let $x =$ the number of years;
and, $y =$ the principal;
then, $y \times \frac{6x}{100} = 720;$ (1)
$y \times \frac{6(x+3)}{100} = 1800.$ (2)
Clearing, $6xy = 72000;$ (3)
$6xy + 18y = 180000;$ (4)
(4) − (3) $18y = 108000;$ (5)
whence, $y = 6000$ dollars, principal;
and, $x = 2$, number of years.

31. Let $x =$ the length;
$y =$ the breadth;
then, $(x+1)(y+1) = xy + 14;$ (1)
and, $(x+1)(y+\frac{1}{2}) = xy;$ (2)
whence, $x = 9$ feet, length;
$y = 4$ feet, breadth.

32. Let $x =$ the number in one pocket;
$y =$ the number in other;
then, $x + y = 39;$ (1)
and, $x - 9 = \frac{2y}{3};$ (2)
whence, $x = 21$ marbles;
$y = 18$ marbles.

Page 158.

33. Let x and $y =$ the digits;
then, $10x + y =$ the number;
then, $10x + y - 2 = 8(x+y);$ (1)
and, $10x + y - 54 = 10y + x;$ (2)
whence, $x = 8;$
$y = 2;$
and, $10x + y = 82$, the number.

34. Let $x =$ time in which all can do the work;

$\frac{1}{y} =$ part of work B can do in 1 da.;

$\frac{3}{2y} =$ part of work A can do in 1 da.;

then, $$\frac{1}{6} + \frac{1}{y} = \frac{1}{x}; \qquad (1)$$

and, $$\frac{1}{8} + \frac{3}{2y} = \frac{1}{x}; \qquad (2)$$

whence, $x = 4$, days in which all can do it.

35. Let $x =$ what he invested at 3%;

$y =$ what he invested, at 2½%;

then, $$x + y = 4400; \qquad (1)$$

and, $$\frac{3x}{100} = \frac{5y}{200}; \qquad (2)$$

whence, $x = 2000$ dollars, at 3%;

and, $y = 2400$ dollars, at 2½%.

36. Let $x =$ yd. A runs per minute;

$y =$ yd. B runs per minute;

then, $$4x - 4y = 100; \qquad (1)$$

and, $$\frac{2750}{y} = \frac{3000}{x}; \qquad (2)$$

whence, $x = 300$ yd., A's rate;

$y = 275$ yd., B's rate.

37. Let $5x$ represent the original rate per hour;

and, y the whole distance;

then, $y - 5x =$ the distance to be traveled after detention;

$\frac{y-5x}{5x} =$ the time required to travel this distance at the original rate;

$\frac{y-5x}{6x} =$ the time required to travel this distance at the increased rate.

Since the train arrived only 10 minutes late, although there was a detention of 30 minutes, the running time of the train after the detention must have been ⅓ of an hour less than usual.

Hence, $$\frac{y-5x}{5x}-\frac{y-5x}{6x}=\frac{1}{3}. \qquad (1)$$

If the detention had occurred 12 miles further on, the distance to be traveled after the detention would have been $y - 5x - 12$.

Hence, $$\frac{y-5x-12}{5x}-\frac{y-5x-12}{6x}=\frac{4}{15}. \qquad (2)$$

Subtracting (2) from (1), $$\frac{12}{5x}-\frac{12}{6x}=\frac{1}{15}; \qquad (3)$$

whence, $x = 6$; and $5x = 30$ miles, the rate per hour.

Substituting in (1), $y = 90$ miles, whole distance.

THREE OR MORE UNKNOWN QUANTITIES.

Page 160.

9. (3) − (2), $4y = 5z.$ (5)

(3) − (1), $3x = 5z.$ (6)

(3) − (4), $2y - 4z = -36.$ (7)

Substituting (5) in (7), $2\frac{1}{2}z - 4z = -36;$ (8)

whence, $z = 24.$

Substituting in other equations, $x = 40$; $y = 30$; $u = 26$; $z = 24$

Page 161.

19. (3) + (1) × 3, $-6y - 2z = -4a.$ (4)

(2) + (1) × 2, $-y - 4z = -3a.$ (5)

(4) − (5) × 6, $22z = 14a;$ (6)

whence, $$z = \frac{7a}{11}.$$

Substituting in other equations, $$y = \frac{5a}{11};\quad x = \frac{a}{11}.$$

20. The sum of all the equations is

$$4x + 4y + 4z = 4(a + b + c). \qquad (4)$$

(4) ÷ 4, $x + y + z = a + b + c.$ (5)

(1) − (5), $z = b + c - a.$

(2) − (5), $y = a + c - b.$

(3) − (5), $x = a + b - c.$

21. The sum of all the equations is

$$2ax + 2by + 2cz = r + s + v. \tag{4}$$

(4) − (2) × 2,
$$2ax = r + v - s. \tag{5}$$

$$x = \frac{r + v - s}{2a};$$

(4) − (3) × 2,
$$y = \frac{r + s - v}{2b};$$

(4) − (1) × 2,
$$z = \frac{s + v - r}{2c}.$$

23. The sum of all the equations is

$$4u + 4v + 4x + 4y + 4z = 80. \tag{6}$$

(6) ÷ 4,
$$u + v + x + y + z = 20. \tag{7}$$

(7) − (1), etc., successively, $z = 6$; $y = 5$; $x = 4$; $v = 3$; $u = 2$.

24. The sum of the equations is

$$\frac{2}{x} + \frac{2}{y} + \frac{2}{z} = 18. \tag{4}$$

(4) ÷ 2,
$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 9. \tag{5}$$

Subtracting given equations each from (5), etc.,

$$x = \tfrac{1}{2};\ y = \tfrac{1}{3};\ z = \tfrac{1}{4}.$$

25. Clearing,
$$3x + y = 15; \tag{4}$$
$$3x + z = 18; \tag{5}$$
$$3y + z = 27. \tag{6}$$

(4) + (5) + (6),
$$6x + 4y + 2z = 60. \tag{7}$$

(7) ÷ 2,
$$3x + 2y + z = 30. \tag{8}$$

(8) − (5),
$$2y = 12; \tag{9}$$

whence,
$$y = 6. \tag{10}$$

Substituting in other equations, $x = 3$; $z = 9$.

26. (1) − (2),
$$\frac{y}{b} - \frac{z}{c} = 0. \tag{4}$$

(3) + (4),
$$\frac{2y}{b} = 2; \tag{5}$$

whence,
$$y = b. \tag{6}$$

Substituting in other equations, $x = a$; $z = c$.

27. (1) + (2) + (3), $2x + 2y + 2z = a + b + c.$ (4)

(1) × 2, $2x + 2y = 2a.$ (5)

(4) − (5), $2z = b + c - a;$ (6)

whence, $z = \frac{b + c - a}{2}.$ (7)

Similarly the other values are found:

$$y = \frac{a - b + c}{2};\quad x = \frac{a + b - c}{2}.$$

28. (2) − (1), $z - x = 2.$ (6)

(3) − (6), $w + x = 11.$ (7)

(4) − (7), $u - x = 4.$ (8)

(5) + (8), $2u = 16;$ (9)

whence, $u = 8.$ (10)

Substituting in other equations, $x = 4$; $y = 5$; $z = 6$; $w = 7$.

29. Clearing, $5xy = x + y;$ (4)

$6yz = x + z;$ (5)

$7xz = x + z.$ (6)

From (5) and (6), $6yz = 7xz;$ (7)

whence, $y = \frac{7x}{6}.$ (8)

Substituting in other equations, $x = \frac{13}{35}$; $y = \frac{13}{30}$; $z = \frac{13}{56}$.

Page 162.

30. (1) − (2), $\frac{1}{y} - \frac{1}{z} = a - b.$ (4)

(4) + (3), $\frac{2}{y} = a - b + c;$ (5)

whence, $y = \frac{2}{a - b + c}.$

Substituting, $x = \frac{2}{a + b - c};\quad z = \frac{2}{b - a + c}.$

31. (3) + (5), $2x + 3t = 30.$ (6)

(6) + (2), $2x + 2y + 3z = 34.$ (7)

(1) × 3, $9x + 12y + 3z = 105.$ (8)

(8) − (7), $7x + 10y = 71.$ (9)

(5) × 3, $3t + 3y = 39.$ (10)

(10) + (2), $3z + 5y = 43.$ (11)

(1) × 3, $9x + 12y + 3z = 105.$ (12)

(12) − (11), $9x + 7y = 62.$ (13)

From (9) and (13), $x = 3;\ y = 5.$

Substituting in other equations, $z = 6;\ u = 7;\ t = 8.$

32. (1) + (2), $x + z = 4a.$ (4)

(3) + (4), $6z = 6a;$ (5)

whence, $z = a.$

Substituting, $y = 2a;\ x = 3a.$

33. (1) + (2), $2y = a + b;$ (4)

whence, $y = \frac{a+b}{2}.$

(2) + (3) $2x = b + c;$ (5)

whence, $x = \frac{b+c}{2}.$

(1) + (3) $2z = a + c;$ (6)

whence, $z = \frac{a+c}{2}.$

34. (1) − (2), $x - 2z + v = a - b.$ (5)

(5) + (3), $2v - z = a - b + c.$ (6)

(3) + (4), $2v - y + z = c + d.$ (7)

(7) + (2), $v + 2z = b + c + d.$ (8)

(8) + (6) × 2, $5v = 2a - b + 3c + d;$ (9)

whence, $v = \frac{2a - b + 3c + d}{5}.$

Substituting in (8), $z = \frac{3b - a + c + 2d}{5}.$

Substituting in (7), $y = \frac{3a + b + 2c - d}{5}.$

Substituting in (5), $x = \frac{a + 2b - c + 3d}{5}.$

35. (1) − (2), $\frac{2c}{z} = a - b;$ (4)

whence, $z = \frac{2c}{a-b}.$

(2) − (3), $\frac{2b}{y} = b - c;$ (5)

whence, $y = \frac{2b}{b-c}.$

(1) + (3), $\frac{2a}{x} = a + c;$ (6)

whence, $x = \frac{2a}{a+c}.$

36. (1) − (2) × 2, $x - 4z = a - 2b.$ (4)

(3) − (4), $6z = 2b - a + c;$ (5)

whence, $z = \frac{2b - a + c}{6}.$

Substituting, $y = \frac{a + b - c}{3};$

$x = \frac{a - 2b + 2c}{3}.$

37. (1) − (2) × 3, $5y + 11z = 3b - a.$ (4)

(2) × 4 − (3), $11y + 13z = 4b - c.$ (5)

(4) × 11 − (5) × 5, $56z = 13b - 11a + 5c;$ (6)

whence, $z = \frac{13b - 11a + 5c}{56}.$

Substituting, $y = \frac{5b + 13a - 11c}{56};$

$x = \frac{5a - 11b + 13c}{56}.$

38. (1) − (2), $x - z = a - c.$ (5)

(5) + (3), $x + v = 2a - b - c.$ (6)

(6) + (4), $2v = 2a - 2b;$ (7)

whence, $v = a - b.$

Substituting in (6), $x = a - c.$

Substituting in (1), $y = b + c;\ z = 0.$

39. (1) + (2), $\frac{2b}{y} = b(a + c);$

whence, $y = \frac{2}{a + c}.$

(2) + (3), $\frac{2c}{z} = c(a + b);$

whence, $z = \frac{2}{a + b}.$

(1) + (3), $\frac{2a}{x} = a(b + c);$

whence, $x = \frac{2}{b + c}.$

THREE OR MORE UNKNOWN QUANTITIES—PROBLEMS.

Page 162.

1. Let x, y, and z represent the numbers;

then, $x + y + z = 60;$ (1)

$\frac{x}{2} + \frac{y}{3} + \frac{z}{5} = 19;$ (2)

$2x + 3y - 3z = 50.$ (3)

Solving, $x = 10;\ y = 30;\ z = 20.$

2. Let x, y, and z represent the numbers;

then, $x + \frac{y}{2} + \frac{z}{2} = 119;$ (1)

$y + \frac{z}{3} - \frac{x}{3} = 68;$ (2)

$\frac{x}{2} + \frac{y}{2} + \frac{z}{2} = 94.$ (3)

Solving, $x = 50;\ y = 58;\ z = 80.$

Page 163.

3. Let x represent A's money, y B's money, and z C's money;

then, $x + y + z = 1500;$ (1)

$x + 200 = y - 200 + 280;$ (2)

$y + 180 = z - 180.$ (3)

Solving, $x = 300$ dollars; $y = 420$ dollars; $z = 780$ dollars.

4. Let x, y, and z represent respectively the prices of the sugar, coffee, and tea per pound;

then, $$7x + 5y + 3z = 4.20; \qquad (1)$$
$$9x + 4y + 2z = 3.40; \qquad (2)$$
$$5x + 2y + 3z = 3.25. \qquad (3)$$

Solving, $x = \$.10$; $y = \$.25$; $z = \$.75$.

5. Let x, y, z, and w represent the parts;

then, $$x + y + z + w = 125; \qquad (1)$$
$$x + 4 = y - 4; \qquad (2)$$
$$y - 4 = 4z; \qquad (3)$$
$$4z = \frac{w}{4}. \qquad (4)$$

Transposing in (2), $$x - y = -8. \qquad (5)$$

(1) − (5), $$2y + z + w = 133. \qquad (6)$$

From (3), $$y = 4z + 4. \qquad (7)$$

Substituting in (6), $$8z + 8 + z + w = 133. \qquad (8)$$

Transposing, etc., $$9z + w = 125. \qquad (9)$$

From (4), $$w = 16z. \qquad (10)$$

Substituting in (9), $$9z + 16z = 125;$$

whence, $$z = 5.$$

Substituting in other equations, $x = 16$; $y = 24$; $w = 80$.

6. Let x, y, and z represent the number of days in which A, B, and C respectively can do the work;

then, $\frac{1}{x}$ = the part A can do in one day;

$\frac{1}{y}$ = the part B can do in one day;

$\frac{1}{z}$ = the part C can do in one day;

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{8}; \qquad (1)$$
$$\frac{1}{x} + \frac{1}{z} = \frac{1}{9}; \qquad (2)$$
$$\frac{1}{y} + \frac{1}{z} = \frac{1}{10}. \qquad (3)$$

(1) + (2) + (3), $\frac{2}{x}+\frac{2}{y}+\frac{2}{z}=\frac{121}{360}.$ (4)

(4) ÷ 2, $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{121}{720}.$ (5)

(5) − (1), $\frac{1}{z}=\frac{31}{720};$ (6)

whence, $z = 23\frac{7}{31}$ da.

(5) − (2), $y = 17\frac{23}{41}$ da.

(5) − (3), $x = 14\frac{34}{49}$ da.

7. Let x, y, and z represent the digits;

then, $100x + 10y + z =$ the number;

$x + y + z = 10;$ (1)

$x + z = \frac{2y}{3};$ (2)

$100x + 10y + z - 198 = x + 10y + 100z.$ (3)

Transposing (3), $99x - 99z = 198;$ (4)

whence, $x - z = 2.$ (5)

(1) − (2), $\frac{5}{3}y = 10;$ (6)

whence, $y = 6.$ (7)

Substituting, $x = 3;$

$z = 1;$ and the number is 361.

8. Let $x =$ the number of horses;

$y =$ the number of cows;

$z =$ the number of sheep;

then, $x + y + z = 80;$ (1)

$\frac{x}{4}+\frac{y}{2}+\frac{z}{6}=19;$ (2)

$\frac{z}{2} - x - y = 10.$ (3)

(2) becomes, $3x + 6y + 2z = 228.$ (4)

(3) becomes, $z - 2x - 2y = 20.$ (5)

(1) × 2 − (2), $-x - 4y = -68.$ (6)

(1) − (3), $3x + 3y = 60.$ (7)

(5) + (4) × 3, $-9y = -144;$

$y = 16$ cows.

Substituting, $x = 4$ horses;

$z = 60$ sheep.

9. Let $x =$ the number of dollar pieces;

$y =$ the number of half-dollar pieces;

$z =$ the number of quarter pieces;

then, $x + y + z = 84;$ (1)

$x + \frac{y}{2} + \frac{z}{4} = 42;$ (2)

$\frac{y}{6} + \frac{z}{16} = 6.50.$ (3)

(2) becomes, $4x + 2y + z = 168.$ (4)

(3) becomes, $8y + 3z = 312;$ (5)

(1) × 4 − (4), $2y + 3z = 168.$ (6)

(6) × 4 − (5), $9z = 360;$

$z = 40$ quarters.

Substituting, $y = 24$ half-dollars;

$x = 20$ dollars.

Page 164.

10. Let $x =$ A's age; $y =$ B's age; $z =$ C's age;

then, $2x + y = z + 20;$ (1)

$\frac{x}{2} + 2y = z - 20;$ (2)

$x + z = 6y.$ (3)

(1) becomes, $2x + y - z = 20.$ (4)

(2) becomes, $x + 4y - 2z = -40.$ (5)

(3) becomes, $x - 6y + z = 0.$ (6)

(4) × 2 − (5), $3x - 2y = 80.$ (7)

(5) + (6) × 2, $3x - 8y = -40.$ (8)

(7) − (8), $6y = 120;$

whence, $y = 20$, B's.

Substituting, $x = 40$, A's; $z = 80$, C's.

11. Let $x =$ the value of fine gold watch;

$y =$ the value of plain gold watch;

$z =$ the value of open-face watch;

$u =$ the value of chain;

then, $x + y + z + u = 400;$ (1)

$x + u = y + z + 40;$ (2)

$y + u = 2z + 20;$ (3)

$z + u = \frac{x}{5} + \frac{y}{3}.$ (4)

These equations become,

$x + y + z + u = 400;$ (1)

$x - y - z + u = 40;$ (2)

$y - 2z + u = 20;$ (3)

$-3x - 5y + 15z + 15u = 0.$ (4)

(1) − (2), $2y + 2z = 360;$ (5)

(2) × 3 + (4), $-8y + 12z + 18u = 120;$ (6)

(3) × 9 − (6) ÷ 2, $13y - 24z = 120;$ (7)

(7) + (5) × 12, $37y = 4440;$ (8)

whence, $y = 120$ dollars.

Substituting, $z = 60$ dollars; $u = 20$ dollars; $x = 200$ dollars.

12. Let $x =$ the cost of a horse;

$y =$ the cost of a sheep;

$z =$ the number of cows;

then, $8x + 100y + 40z = 2500;$ (1)

$z = 4y;$ (2)

$x + y = 8z - 5.$ (3)

(1) − (3) × 8, $104z + 92y = 2540.$ (4)

(4) + (2) × 23, $127z = 2540;$ (5)

whence, $z = 20$, the number of cows.

Substituting, $y = 5$ dollars, the cost of a sheep;

$x = 150$ dollars, the cost of a horse.

13. Let $x =$ A's nuts; $y =$ B's nuts; $z =$ C's nuts;

then,

$$x - \frac{2x}{5} + \frac{y}{5} + \frac{z}{5} = 740;$$

$$y - \frac{2y}{5} + \frac{x}{5} + \frac{z}{5} = 580;$$

$$z - \frac{2z}{5} + \frac{x}{5} + \frac{y}{5} = 380.$$

These equations become,

$$3x + y + z = 3700; \tag{1}$$

$$x + 3y + z = 2900; \tag{2}$$

$$x + y + 3z = 1900. \tag{3}$$

(1) + (2) + (3), $$5x + 5y + 5z = 8500; \tag{4}$$

(4) ÷ 5, $$x + y + z = 1700. \tag{5}$$

(1) − (5), $$2x = 2000;$$

whence, $$x = 1000 \text{ nuts, A's.}$$

(2) − (5), $$2y = 1200.$$

whence, $$y = 600 \text{ nuts, B's.}$$

(3) − (5), $$2z = 200;$$

whence, $$z = 100 \text{ nuts, C's.}$$

14. Let $\frac{x}{y}$ and $\frac{z}{y}$ represent the fractions;

then,

$$\frac{x-1}{y} = \frac{z}{3y}; \tag{1}$$

$$\frac{2x}{y} = \frac{z-1}{y}; \tag{2}$$

$$\frac{x}{y} - \frac{z}{y} = \frac{1}{3}. \tag{3}$$

Clearing, etc. (1), $$3x - z = 3. \tag{4}$$

From (2), $$2x - z = -1. \tag{5}$$

From (4) and (5), $$x = 4;\ z = 9.$$

Substituting in (3), $$y = 15.$$

Hence, the fractions are $\frac{4}{15}$ and $\frac{9}{15}$.

15. Let x, y, z, and w represent respectively the shares of each;

then,
$$x = \frac{y}{2} + \frac{z}{2} + \frac{w}{2}; \tag{1}$$
$$y = \frac{x}{3} + \frac{z}{3} + \frac{w}{3}; \tag{2}$$
$$z = \frac{x}{4} + \frac{y}{4} + \frac{w}{4}; \tag{3}$$
$$x - w = 14. \tag{4}$$

Clearing, etc.,
$$2x - y - z - w = 0. \tag{5}$$
$$-x + 3y - z - w = 0. \tag{6}$$
$$-x - y + 4z - w = 0. \tag{7}$$

(5) − (4),
$$x - y - z = -14. \tag{8}$$

(6) − (4),
$$-2x + 3y - z = -14. \tag{9}$$

(7) − (4),
$$2x - y + 4z = -14. \tag{10}$$

(8) × 2,
$$2x - 2y - 2z = -28. \tag{11}$$

(9) + (11),
$$y - 3z = -42. \tag{12}$$

(10) + (11),
$$-3y + 2z = -42. \tag{13}$$

From (12) + (13), $y = 30$ dollars; $z = 24$ dollars.

Substituting in other equations, $x = 40$ dollars; $w = 26$ dollars.

Page 165.

16. Let x represent the number of sheep, y the number of cows, and z the number of horses;

then,
$$x - 26 = y + z; \tag{1}$$
$$\frac{x}{5} = \frac{y}{4} + z; \tag{2}$$
$$\frac{x}{5} + \frac{y}{8} + \frac{z}{2} = 12. \tag{3}$$

Clearing, etc., $x = 40$ sheep; $y = 8$ cows; $z = 6$ horses.

17. Let x, y, and z represent the amounts in the respective purses;

then,
$$x - 20 = \frac{y + 20}{4}; \tag{1}$$
$$y - 60 = \frac{z + 60}{1\frac{3}{4}}; \tag{2}$$
$$x + 40 = \frac{z - 40}{2\frac{1}{4}}. \tag{3}$$

These equations become, $4x - y = 100;$ (4)

$7y - 4z = 660;$ (5)

$9x - 4z = -520.$ (6)

$(5)-(6),$ $7y - 9x = 1180;$ (7)

$(7)+(4)\times 7,$ $19x = 1880;$

whence, $x = 98\frac{18}{19}$ dollars.

Substituting, $y = 295\frac{15}{19}$ dollars;

$z = 352\frac{12}{19}$ dollars.

18. Let x, y, and z represent respectively the sums paid by A, B, and C; then,

$$x + y + z = 900; \qquad (1)$$

$$\frac{x}{2} + \frac{y}{4} + \frac{z}{5} = 279; \qquad (2)$$

$$x + \frac{2y}{3} - \frac{z}{2} = 320. \qquad (3)$$

Clearing, etc., $x = 280$ dollars, A's;

$y = 300$ dollars, B's;

$z = 320$ dollars, C's.

19. Let x, y, and z represent the numbers respectively;

then,

$$\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 115; \qquad (1)$$

$$\frac{x}{3} + \frac{y}{4} + \frac{z}{5} = 86; \qquad (2)$$

$$\frac{x}{4} + \frac{y}{5} + \frac{z}{6} = 69. \qquad (3)$$

These equations become,

$$6x + 4y + 3z = 1380; \qquad (4)$$

$$20x + 15y + 12z = 5160; \qquad (5)$$

$$15x + 12y + 10z = 4140. \qquad (6)$$

From the equations by uniting, etc.,

$$x = 60;\ y = 120;\ z = 180.$$

20. Let x = A's money at first; y = B's; z = C's;

then, $x - y - z$ = A's after first giving;

$2y$ = B's after first giving;

$2z$ = C's after first giving;

$$\begin{aligned} 2x - 2y - 2z &= \text{A's after second giving;} \\ 3y - x - z &= \text{B's after second giving;} \\ 4z &= \text{C's after second giving;} \\ 4x - 4y - 4z &= \text{A's after third giving;} \\ 6y - 2x - 2z &= \text{B's after third giving;} \\ 7z - x - y &= \text{C's after third giving.} \end{aligned}$$

Therefore,

$$4x - 4y - 4z = 8; \quad (1)$$

$$-2x + 6y - 2z = 8; \quad (2)$$

$$-x - y + 7z = 8. \quad (3)$$

From these equations by uniting, etc.,

$$x = 13 \text{ dollars}; \; y = 7 \text{ dollars}; \; z = 4 \text{ dollars}.$$

21. Let $x = \text{A's money}; \; y = \text{B's}; \; z = \text{C's}.$

$$\frac{x + 350}{2} = y - 350; \quad (1)$$

$$\frac{y + 700}{3} = z - 700; \quad (2)$$

$$\frac{z + 210}{5} = x - 210. \quad (3)$$

Clearing,

$$x - 2y = -1050; \quad (4)$$

$$y - 3z = -2800; \quad (5)$$

$$-5x + z = -1260. \quad (6)$$

Uniting these equations, etc., $x = 490$ dollars;

$y = 770$ dollars;

$z = 1190$ dollars.

INVOLUTION.

Page 171.

10. $$(x + 1)^3 = x^3 + 3x^2 \cdot 1 + 3x \cdot 1^2 + 1^3.$$

Since every power of 1 is 1, it appears in no term except the last, and the answer becomes,

$$x^3 + 3x^2 + 3x + 1.$$

11. $$x^3 - 3x^2 + 3x - 1.$$

12. $$(1 + a)^3 = 1^3 + 3 \cdot 1^2 \cdot a + 3 \cdot 1 \cdot a^2 + a^3.$$

Since every power of 1 is 1, it appears in no term except the first, and the answer becomes,

$$1 + 3a + 3a^2 + a^3.$$

13. See Ex. 12.

25 and **26**. Raise in the usual way, remembering that one appears in no term except the last.

27 and **28**. Raise in the usual way, remembering that 1 appears in no term except the first.

29, **30**, and **31**. In these examples the second term is composed of two letters which are to be considered as a single quantity.

Page 172.

33. $(a + 2b)^3 = a^3 + 3a^2(2b) + 3a(2b)^2 + (2b)^3$
$= a^3 + 6a^2b + 12ab^2 + 8b^3.$

34. $(3a - b)^3 = (3a)^3 - 3(3a)^2b + 3(3a)b^2 - b^3$
$= 27a^3 - 27a^2b + 9ab^2 - b^3.$

35. $(2a + 3b)^3 = (2a)^3 + 3(2a)^2(3b) + 3(2a)(3b)^2 + (3b)^3$
$= 8a^3 + 36a^2b + 54ab^2 + 27b^3.$

36. $(3a - 3c)^3 = 3^3(a - c)^3 = 27(a^3 - 3a^2c + 3ac^2 - c^3)$
$= 27a^3 - 81a^2c + 81ac^2 - 27c^3.$

37. $(3a + 2c)^4 = (3a)^4 + 4(3a)^3(2c) + 6(3a)^2(2c)^2 + 4(3a)(2c)^3 + (2c)^4$
$= 81a^4 + 216a^3c + 216a^2c^2 + 96ac^3 + 16c^4.$

38. $(2a^2 - c)^3 = (2a^2)^3 - 3(2a^2)^2c + 3(2a^2)c^2 - c^3$
$= 8a^6 - 12a^4c + 6a^2c^2 - c^3.$

39. $(2a + c^2)^3 = (2a)^3 + 3(2a)^2c^2 + 3(2a)(c^2)^2 + (c^2)^3$
$= 8a^3 + 12a^2c^2 + 6ac^4 + c^6.$

40. $(3x^2 + 2y^2)^3 = (3x^2)^3 + 3(3x^2)^2(2y^2) + 3(3x^2)(2y^2)^2 + (2y^2)^3$
$= 27x^6 + 54x^4y^2 + 36x^2y^4 + 8y^6.$

41. $(4a^2 - c)^4 = (4a^2)^4 - 4(4a^2)^3c + 6(4a^2)^2c^2 - 4(4a^2)c^3 + c^4$
$= 256a^8 - 256a^6c + 96a^4c^2 - 16a^2c^3 + c^4.$

42. $(3a^2 + 4c^2)^3 = (3a^2)^3 + 3(3a^2)^2(4c^2) + 3(3a^2)(4c^2)^2 + (4c^2)^3$
$= 27a^6 + 108a^4c^2 + 144a^2c^4 + 64c^6.$

43. $(3x^2 - 5c^2)^4 = (3x^2)^4 - 4(3x^2)^3(5c^2) + 6(3x^2)^2(5c^2)^2 - 4(3x^2)(5c^2)^3$
$+ (5c^2)^4 = 81x^8 - 540x^6c^2 + 1350x^4c^4 - 1500x^2c^6 + 625c^8.$

44. $(5a^2+4c^2)^4=(5a^2)^4+4(5a^2)^3(4c^2)+6(5a^2)^2(4c^2)^2+4(5a^2)(4c^2)^3$
$+(4c^2)^4 = 625a^8 + 2000a^6c^2 + 2400a^4c^4 + 1280a^2c^6 + 256c^8.$

47. $(x-y+z)^3=((x-y)+z)^3=(x-y)^3+3(x-y)^2z+3(x-y)z^2+z^3$
$= x^3 - 3x^2y + 3xy^2 - y^3 + 3x^2z - 6xyz + 3y^2z + 3xz^2 - 3yz^2 + z^3.$

48. $(x-y-z)^3=((x-y)-z)^3=(x-y)^3-3(x-y)^2z+3(x-y)z^2-z^3$
$= x^3 - 3x^2y + 3xy^2 - y^3 - 3x^2z + 6xyz - 3y^2z + 3xz^2 - 3yz^2 - z^3.$

49. $(a+b-c)^3=((a+b)-c)^3=(a+b)^3-3(a+b)^2c+3(a+b)c^2-c^3$
$= a^3 + 3a^2b + 3ab^2 + b^3 - 3a^2c - 6abc - 3b^2c + 3ac^2 + 3bc^2 - c^3.$

50. $(a-b-c-d)^3=((a-b)-(c+d))^3=(a-b)^3-3(a-b)^2(c+d)$
$+3(a-b)(c+d)^2-(c+d)^3 = a^3 - 3a^2b + 3ab^2 - b^3 - 3a^2c$
$+ 6abc - 3b^2c - 3a^2d + 6abd - 3b^2d + 3ac^2 + 6acd + 3ad^2$
$- 3bc^2 - 6bcd - 3bd^2 - c^3 - 3c^2d - 3cd^2 - d^3.$

51. $(a+b-c+d)^3=((a+b)-(c-d))^3=(a+b)^3-3(a+b)^2(c-d)$
$+3(a+b)(c-d)^2-(c-d)^3 = a^3 + 3a^2b + 3ab^2 + b^3 - 3a^2c$
$- 6abc - 3b^2c + 3a^2d + 6abd + 3b^2d + 3ac^2 - 6acd + 3ad^2$
$+ 3bc^2 - 6bcd + 3bd^2 - c^3 + 3c^2d - 3cd^2 + d^3.$

52. $(1+x-y-z)^3=((1+x)-(y+z))^3=(1+x)^3-3(1+x)^2(y+z)$
$+3(1+x)(y+z)^2-(y+z)^3 = 1 + 3x + 3x^2 + x^3 - 3y - 6xy$
$- 3x^2y - 3z - 6xz - 3x^2z + 3y^2 + 6yz + 3z^2 + 3xy^2 + 6xyz$
$+ 3xz^2 - y^3 - 3y^2z - 3yz^2 - z^3.$

Page 174.

3. The coefficients are:

1st, 8; 2d, $\frac{8\times3\times3}{2}=36$; 3d, $\frac{36\times3\times2}{2\times2}=54$; 4th, $\frac{54\times3\times1}{2\times3}=27.$

$\therefore\ (2a+3c)^3 = 8a^3 + 36a^2c + 54ac^2 + 27c^3.$

4. The coefficients are:

1st, 8; 2d, $\frac{8\times4\times3}{2}=48$; 3d, $\frac{48\times4\times2}{2\times2}=96$; 4th, $\frac{96\times4\times1}{2\times3}=64.$

$\therefore\ (2a-4b)^3 = 8a^3 - 48a^2b + 96ab^2 - 64b^3.$

5. The coefficients are:

1st, 81; 2d, $\frac{81\times2\times4}{3}=216$; 3d, $\frac{216\times2\times3}{3\times2}=216$;

4th, $\frac{216\times2\times2}{3\times3}=96$; 5th, $\frac{96\times2\times1}{3\times4}=16.$

$\therefore\ (3a+2x)^4 = 81a^4 + 216a^3x + 216a^2x^2 + 96ax^3 + 16x^4.$

6. The coefficients are:

1st, 32; 2d, $\frac{32\times5\times5}{2}=400$; 3d, $\frac{400\times5\times4}{2\times2}=2000$;

4th, $\frac{2000\times5\times3}{2\times3}=5000$; 5th, $\frac{5000\times5\times2}{2\times4}=6250$; 6th, $\frac{6250\times5\times1}{2\times5}=3125$.

$\therefore\ (2a-5x)^5=32a^5-400a^4x+2000a^3x^2-5000a^2x^3+6250ax^4-3125x^5.$

7. The coefficients are:

1st, 625; 2d, $\frac{625\times3\times4}{5}=1500$; 3d, $\frac{1500\times3\times3}{5\times2}=1350$;

4th, $\frac{1350\times3\times2}{5\times3}=540$; 5th, $\frac{540\times3\times1}{5\times4}=81$.

$\therefore\ (5a+3c)^4=625a^4+1500a^3c+1350a^2c^2+540ac^3+81c^4.$

8. The coefficients are:

1st, 1024; 2d, $\frac{1024\times3\times5}{4}=3840$; 3d, $\frac{3840\times3\times4}{4\times2}=5760$;

4th, $\frac{5760\times3\times3}{4\times3}=4320$; 5th, $\frac{4320\times3\times2}{4\times4}=1620$; 6th, $\frac{1620\times3\times1}{4\times5}=243$.

$\therefore\ (4a+3x)^5=1024a^5+3840a^4x+5760a^3x^2+4320a^2x^3+1620ax^4+243x^5.$

9. The coefficients are:

1st, 243; 2d, $\frac{243\times5\times5}{3}=2025$; 3d, $\frac{2025\times5\times4}{3\times2}=6750$;

4th, $\frac{6750\times5\times3}{3\times3}=11250$; 5th, $\frac{11250\times5\times2}{3\times4}=9375$; 6th, $\frac{9375\times5\times1}{3\times5}=3125$.

$\therefore\ (3a-5x)^5=243a^5-2025a^4x+6750a^3x^2-11250a^2x^3+9375ax^4-3125x^5.$

10. The coefficients are:

1st, 2401; 2d, $\frac{2401\times4\times4}{7}=5488$; 3d, $\frac{5488\times4\times3}{7\times2}=4704$;

4th, $\frac{4704\times4\times2}{7\times3}=1792$; 5th, $\frac{1792\times4\times1}{7\times4}=256$.

$\therefore\ (7a-4c)^4=2401a^4-5488a^3c+4704a^2c^2-1792ac^3+256c^4.$

11. The coefficients are:

1st, 46656; 2d, $\frac{46656\times5\times6}{6}=233280$; 3d, $\frac{233280\times5\times5}{6\times2}=486000$;

4th, $\frac{486000\times5\times4}{6\times3}=540000$; 5th, $\frac{540000\times5\times3}{6\times4}=337500$;

6th, $\frac{337500\times5\times2}{6\times5}=112500$; 7th, $\frac{112500\times5\times1}{6\times6}=15625$.

$$\therefore\ (6a+5x)^6=46656a^6+233280a^5x+486000a^4x^2+540000a^3x^3$$
$$+337500a^2x^4+112500ax^5+15625x^6.$$

12. The coefficients are:

1st, 243; 2d, $\frac{243 \times 1 \times 5}{3} = 405$; 3d, $\frac{405 \times 1 \times 4}{3 \times 2} = 270$;

4th, $\frac{270 \times 1 \times 3}{3 \times 3} = 90$; 5th, $\frac{90 \times 1 \times 2}{3 \times 4} = 15$; 6th, $\frac{15 \times 1 \times 1}{3 \times 5} = 1$.

$\therefore (3x - 1)^5 = 243x^5 - 405x^4 + 270x^3 - 90x^2 + 15x - 1.$

13. The coefficients are:

1st, 16; 2d, $\frac{16 \times 5 \times 4}{2} = 160$; 3d, $\frac{160 \times 5 \times 3}{2 \times 2} = 600$;

4th, $\frac{600 \times 5 \times 2}{2 \times 3} = 1000$; 5th, $\frac{1000 \times 5 \times 1}{2 \times 4} = 625$.

$\therefore (2x + 5)^4 = 16x^4 + 160x^3 + 600x^2 + 1000x + 625.$

14. $\left(a + \frac{1}{2a}\right)^5 = a^5 + 5a^4\left(\frac{1}{2a}\right) + 10a^3\left(\frac{1}{2a}\right)^2 + 10a^2\left(\frac{1}{2a}\right)^3 + 5a\left(\frac{1}{2a}\right)^4$

$+ \left(\frac{1}{2a}\right)^5 = a^5 + \frac{5}{2}a^3 + \frac{5}{2}a + \frac{5}{4a} + \frac{5}{16a^3} + \frac{1}{32a^5}.$

15. The coefficients are:

1st, 1; 2d, $\frac{1 \times \frac{3}{2} \times 5}{1} = \frac{15}{2}$; 3d, $\frac{\frac{15}{2} \times \frac{3}{2} \times 4}{1 \times 2} = \frac{45}{2}$; 4th, $\frac{\frac{45}{2} \times \frac{3}{2} \times 3}{1 \times 3} = \frac{135}{4}$;

5th, $\frac{\frac{135}{4} \times \frac{3}{2} \times 2}{1 \times 4} = \frac{405}{16}$; 6th, $\frac{\frac{405}{16} \times \frac{3}{2} \times 1}{1 \times 5} = \frac{243}{32}$.

$\therefore (1 - \frac{3}{2}x)^5 = 1 - \frac{15}{2}x + \frac{45}{2}x^2 - \frac{135}{4}x^3 + \frac{405}{16}x^4 - \frac{243}{32}x^5.$

16. The coefficients are:

1st, $\frac{81}{16}$; 2d, $\frac{\frac{81}{16} \times \frac{5}{3} \times 4}{\frac{3}{2}} = \frac{45}{2}$; 3d, $\frac{\frac{45}{2} \times \frac{5}{3} \times 3}{\frac{3}{2} \times 2} = \frac{75}{2}$;

4th, $\frac{\frac{75}{2} \times \frac{5}{3} \times 2}{\frac{3}{2} \times 3} = \frac{250}{9}$; 5th, $\frac{\frac{250}{9} \times \frac{5}{3} \times 1}{\frac{3}{2} \times 4} = \frac{625}{81}$.

$\therefore (\frac{3}{2} + \frac{5}{3}c)^4 = \frac{81}{16} + \frac{45}{2}c + \frac{75}{2}c^2 + \frac{250}{9}c^3 + \frac{625}{81}c^4.$

17. The coefficients are:

1st, 1; 2d, $\frac{1 \times 2 \times 5}{1} = 10$; 3d, $\frac{10 \times 2 \times 4}{1 \times 2} = 40$;

4th, $\frac{40 \times 2 \times 3}{1 \times 3} = 80$; 5th, $\frac{80 \times 2 \times 2}{1 \times 4} = 80$; 6th, $\frac{80 \times 2 \times 1}{1 \times 5} = 32$.

$\therefore (a^2 + 2c^2)^5 = a^{10} + 10a^8c^2 + 40a^6c^4 + 80a^4c^6 + 80a^2c^8 + 32c^{10}.$

SQUARE ROOT.

Page 179.

3.

$$\begin{array}{r|l}
\multicolumn{2}{l}{x^2+4x+4\,\underline{|\,x+2}} \\
\multicolumn{2}{l}{x^2} \\ \hline
2x+2 & 4x+4 \\
 & \underline{4x+4}
\end{array}$$

5.

$$\begin{array}{r|l}
\multicolumn{2}{l}{4x^2+4x+1\,\underline{|\,2x+1}} \\
\multicolumn{2}{l}{4x^2} \\ \hline
4x+1 & 4x+1 \\
 & \underline{4x+1}
\end{array}$$

4.

$$\begin{array}{r|l}
\multicolumn{2}{l}{b^2+2bx+x^2\,\underline{|\,b+x}} \\
\multicolumn{2}{l}{b^2} \\ \hline
2b+x & 2bx+x^2 \\
 & \underline{2bx+x^2}
\end{array}$$

6.

$$\begin{array}{r|l}
\multicolumn{2}{l}{a^2+ab+\tfrac{1}{4}b^2\,\underline{|\,a+\tfrac{1}{2}b}} \\
\multicolumn{2}{l}{a^2} \\ \hline
2a+\tfrac{1}{2}b & ab+\tfrac{1}{4}b^2 \\
 & \underline{ab+\tfrac{1}{4}b^2}
\end{array}$$

7.

$$\begin{array}{r|l}
\multicolumn{2}{l}{9a^2-12ab+4b^2\,\underline{|\,3a-2b}} \\
\multicolumn{2}{l}{9a^2} \\ \hline
6a-2b & -12ab+4b^2 \\
 & \underline{-12ab+4b^2}
\end{array}$$

8.

$$\begin{array}{r|l}
\multicolumn{2}{l}{a^2+2ab+b^2-2ac-2bc+c^2\,\underline{|\,a+b-c}} \\
\multicolumn{2}{l}{a^2} \\ \hline
2a+b & 2ab+b^2 \\
 & 2ab+b^2 \\ \hline
2a+2b-c & -2ac-2bc+c^2 \\
 & \underline{-2ac-2bc+c^2}
\end{array}$$

9

$$\begin{array}{r|l}
\multicolumn{2}{l}{4x^4-12x^3+13x^2-6x+1\,\underline{|\,2x^2-3x+1}} \\
\multicolumn{2}{l}{4x^4} \\ \hline
4x^2-3x & -12x^3+13x^2 \\
 & -12x^3+9x^2 \\ \hline
4x^2-6x+1 & 4x^2-6x+1 \\
 & \underline{4x^2-6x+1}
\end{array}$$

10.

$$\begin{array}{r|l}
\multicolumn{2}{l}{4a^4+4a^3-7a^2-4a+4\,\underline{|\,2a^2+a-2}} \\
\multicolumn{2}{l}{4a^4} \\ \hline
4a^2+a & 4a^3-7a^2 \\
 & 4a^3+a^2 \\ \hline
4a^2+2a-2 & -8a^2-4a+4 \\
 & \underline{-8a^2-4a+4}
\end{array}$$

11.

$$\begin{array}{r|l}
\multicolumn{2}{l}{x^6-4x^5+10x^4-12x^3+9x^2\,\underline{|\,x^3-2x^2+3x}} \\
\multicolumn{2}{l}{x^6} \\ \hline
x^3-2x^2 & -4x^5+10x^4 \\
 & -4x^5+4x^4 \\ \hline
2x^3-4x^2+3x & 6x^4-12x^3+9x^2 \\
 & \underline{6x^4-12x^3+9x^2}
\end{array}$$

12.

$$\begin{array}{r|l}
\multicolumn{2}{r}{16a^4 - 24a^3x + 49a^2x^2 - 30ax^3 + 25x^4 \,\underline{|\,4a^2 - 3ax + 5x^2}} \\
\multicolumn{2}{l}{16a^4} \\ \hline
8a^2 - 3ax & -24a^3x + 49a^2x^2 \\
 & -24a^3x + 9a^2x^2 \\ \hline
8a^2 - 6ax + 5x^2 & 40a^2x^2 - 30ax^3 + 25x^4 \\
 & 40a^2x^2 - 30ax^3 + 25x^4 \\ \hline
\end{array}$$

13. Before attempting to extract the root, the terms of the polynomial should be arranged with reference to the powers of some letter. Thus:

$$\begin{array}{r|l}
\multicolumn{2}{r}{a^2 - 2ab + b^2 - 2ac + 2bc + c^2 \,\underline{|\,a - b - c}} \\
\multicolumn{2}{l}{a^2} \\ \hline
2a - b & -2ab + b^2 \\
 & -2ab + b^2 \\ \hline
2a - 2b - c & -2ac + 2bc + c^2 \\
 & -2ac + 2bc + c^2 \\ \hline
\end{array}$$

14.

$$\begin{array}{r|l}
\multicolumn{2}{r}{9x^4 - 12x^3 + 40x^2 - 24x + 36 \,\underline{|\,3x^2 - 2x + 6}} \\
\multicolumn{2}{l}{9x^4} \\ \hline
6x^2 - 2x & -12x^3 + 40x^2 \\
 & -12x^3 + 4x^2 \\ \hline
6x^2 - 4x + 6 & 36x^2 - 24x + 36 \\
 & 36x^2 - 24x + 36 \\ \hline
\end{array}$$

15.

$$\begin{array}{r|l}
\multicolumn{2}{r}{4x^6 + 12x^5 + 5x^4 - 10x^3 - 5x^2 + 2x + 1 \,\underline{|\,2x^3 + 3x^2 - x - 1}} \\
\multicolumn{2}{l}{4x^6} \\ \hline
4x^3 + 3x^2 & 12x^5 + 5x^4 \\
 & 12x^5 + 9x^4 \\ \hline
4x^3 + 6x^2 - x & -4x^4 - 10x^3 - 5x^2 \\
 & -4x^4 - 6x^3 + x^2 \\ \hline
4x^3 + 6x^2 - 2x - 1 & -4x^3 - 6x^2 + 2x + 1 \\
 & -4x^3 - 6x^2 + 2x + 1 \\ \hline
\end{array}$$

16.

$$\begin{array}{r|l}
\multicolumn{2}{r}{49x^4 - 28x^3 - 17x^2 + 6x + \tfrac{9}{4} \,\underline{|\,7x^2 - 2x - \tfrac{3}{2}}} \\
\multicolumn{2}{l}{49x^4} \\ \hline
14x^2 - 2x & -28x^3 - 17x^2 \\
 & -28x^3 + 4x^2 \\ \hline
14x^2 - 4x - \tfrac{3}{2} & -21x^2 + 6x + \tfrac{9}{4} \\
 & -21x^2 + 6x + \tfrac{9}{4} \\ \hline
\end{array}$$

17.

$$\begin{array}{r|l}
\multicolumn{2}{r}{9a^2 + 6ac + c^2 - 6a - 2c + 1 \,\underline{|\,3a + c - 1}} \\
\multicolumn{2}{l}{9a^2} \\ \hline
6a + c & 6ac + c^2 \\
 & 6ac + c^2 \\ \hline
6a + 2c - 1 & -6a - 2c + 1 \\
 & -6a - 2c + 1 \\ \hline
\end{array}$$

Page 180.

18.

$$\begin{array}{r|l}
\multicolumn{2}{l}{4m^2 - 12mx + 9x^2 + 8m - 12x + 4 \,\underline{|\,2m - 3x + 2}} \\
\multicolumn{2}{l}{4m^2} \\ \hline
4m - 3x & -12mx + 9x^2 \\
 & -12mx + 9x^2 \\ \hline
4m - 6x + 2 & 8m - 12x + 4 \\
 & 8m - 12x + 4 \\ \hline
\end{array}$$

19.

$$\begin{array}{r|l}
\multicolumn{2}{l}{a^2 - 2ab + b^2 - 12ac + 12bc + 36c^2 \,\underline{|\,a - b - 6c}} \\
\multicolumn{2}{l}{a^2} \\ \hline
2a - b & -2ab + b^2 \\
 & -2ab + b^2 \\ \hline
2a - 2b - 6c & -12ac + 12bc + 36c^2 \\
 & -12ac + 12bc + 36c^2 \\ \hline
\end{array}$$

20.

$$\begin{array}{r|l}
\multicolumn{2}{l}{a^{2m} - 2a^mb^2 + b^4 + 2a^mc^{3n} - 2b^2c^{3n} + c^{6n} \,\underline{|\,a^m - b^2 + c^{3n}}} \\
\multicolumn{2}{l}{a^{2m}} \\ \hline
2a^m - b^2 & -2a^mb^2 + b^4 \\
 & -2a^mb^2 + b^4 \\ \hline
2a^m - 2b^2 + c^{3n} & 2a^mc^{3n} - 2b^2c^{3n} + c^{6n} \\
 & 2a^mc^{3n} - 2b^2c^{3n} + c^{6n} \\ \hline
\end{array}$$

21.

$$\begin{array}{r|l}
\multicolumn{2}{l}{1 - a + \frac{a^2}{4} + \frac{2a}{3} - \frac{a^2}{3} + \frac{a^2}{9} \,\underline{\Big|\,1 - \frac{a}{2} + \frac{a}{3}}} \\
\multicolumn{2}{l}{1} \\ \hline
2 - \frac{a}{2} & -a + \frac{a^2}{4} \\
 & -a + \frac{a^2}{4} \\ \hline
2 - a + \frac{a}{3} & \frac{2a}{3} - \frac{a^2}{3} + \frac{a^2}{9} \\
 & \frac{2a}{3} - \frac{a^2}{3} + \frac{a^2}{9} \\ \hline
\end{array}$$

22.

$$\begin{array}{r|l}
\multicolumn{2}{l}{m^6 - 2m^5 - m^4 + 3m^2 + 2m + 1 \,\underline{|\,m^3 - m^2 - m - 1}} \\
\multicolumn{2}{l}{m^6} \\ \hline
2m^3 - m^2 & -2m^5 - m^4 \\
 & -2m^5 + m^4 \\ \hline
2m^3 - 2m^2 - m & -2m^4 + 3m^2 + 2m \\
 & -2m^4 + m^2 + 2m^3 \\ \hline
2m^3 - 2m^2 - 2m - 1 & -2m^3 + 2m^2 + 2m + 1 \\
 & -2m^3 + 2m^2 + 2m + 1 \\ \hline
\end{array}$$

23.

$$\begin{array}{r|l}
 & a^4 + a^3 + \frac{3}{4}a^2 + \frac{1}{4}a + \frac{1}{16} \;\underline{\Big|\; a^2 + \frac{a}{2} + \frac{1}{4}} \\
 & a^4 \\ \hline
2a^2 + \frac{a}{2} & a^3 + \frac{3}{4}a^2 \\
 & a^3 + \frac{1}{4}a^2 \\ \hline
2a^2 + a + \frac{1}{4} & \frac{2}{4}a^2 + \frac{1}{4}a + \frac{1}{16} \\
 & \frac{2}{4}a^2 + \frac{1}{4}a + \frac{1}{16} \\ \hline
\end{array}$$

24.

$$\begin{array}{r|l}
 & c^4 + 2c^3 + \frac{3}{2}c^2 - \frac{c}{2} + \frac{1}{16} \;\underline{\Big|\; c^2 - c + \frac{1}{4}} \\
 & c^4 \\ \hline
2c^2 - c & -2c^3 + \frac{3c^2}{2} \\
 & -2c^3 + c^2 \\ \hline
2c^2 - 2c + \frac{1}{4} & \frac{1}{2}c^2 - \frac{c}{2} + \frac{1}{16} \\
 & \frac{1}{2}c^2 - \frac{c}{2} + \frac{1}{16} \\ \hline
\end{array}$$

25.

$$\begin{array}{r|l}
 & \frac{c^4}{4} + \frac{c^3}{x} + \frac{c^2}{x^2} - cx + \frac{x^2}{c^2} - 2 \;\underline{\Big|\; \frac{c^2}{2} + \frac{c}{x} - \frac{x}{c}} \\
 & \frac{c^4}{4} \\ \hline
c^2 + \frac{c}{x} & \frac{c^3}{x} + \frac{c^2}{x^2} \\
 & \frac{c^3}{x} + \frac{c^2}{x^2} \\ \hline
c^2 + \frac{2c}{x} - \frac{x}{c} & -cx + \frac{x^2}{c^2} - 2 \\
 & -cx + \frac{x^2}{c^2} - 2 \\ \hline
\end{array}$$

27.

$$\begin{array}{r|l}
 & 1 - x \;\underline{\Big|\; 1 - \frac{x}{2} - \frac{x^2}{8}} \\
 & 1 \\ \hline
2 - \frac{x}{2} & -x \\
 & -x + \frac{x^2}{4} \\ \hline
2 - x - \frac{x^2}{8} & -\frac{x^2}{4} \\
 & -\frac{x^2}{4} + \frac{x^3}{8} + \frac{x^4}{64} \\ \hline
\end{array}$$

28.

$$\begin{array}{r|l}
 & 1-2a \;\big|\; \underline{1-a-\frac{a^2}{2}} \\
 & 1 \\ \hline
2-a & -2a \\
 & -2a+a^2 \\ \hline
2-2a-\frac{a^2}{2} & -a^2 \\
 & -a^2+a^3+a^4 \\ \hline
\end{array}$$

29.

$$\begin{array}{r|l}
 & a^2-2b \;\big|\; \underline{a-\frac{b}{a}-\frac{b^2}{2a^3}} \\
 & a^2 \\ \hline
2a-\frac{b}{a} & -2b \\
 & -2b+\frac{b^2}{a^2} \\ \hline
2a-\frac{2b}{a}-\frac{b^2}{2a^3} & -\frac{b^2}{a^2} \\
 & -\frac{b^2}{a^2}+\frac{b^3}{a^4}+\frac{b^4}{4a^6} \\ \hline
\end{array}$$

30.

$$\begin{array}{r|l}
 & a^2+b \;\big|\; \underline{a+\frac{b}{2a}-\frac{b^2}{8a^3}} \\
 & a^2 \\ \hline
2a+\frac{b}{2a} & b \\
 & b+\frac{b^2}{4a^2} \\ \hline
2a+\frac{b}{a}-\frac{b^2}{8a^3} & -\frac{b^2}{4a^2} \\
 & -\frac{b^2}{4a^2}-\frac{b^3}{8a^4}+\frac{b^4}{64a^6} \\ \hline
\end{array}$$

31.

$$\begin{array}{r|l}
 & 4+3a \;\big|\; \underline{2+\frac{3a}{4}-\frac{9a^2}{64}} \\
 & 4 \\ \hline
4+\frac{3a}{4} & 3a \\
 & 3a+\frac{9a^2}{16} \\ \hline
4+\frac{3a}{2}-\frac{9a^2}{64} & -\frac{9a^2}{16} \\
 & -\frac{9a^2}{16}-\frac{27a^3}{128}+\frac{81a^4}{4096} \\ \hline
\end{array}$$

32.

$$\begin{array}{r|l}
 & 4a^2+3 \;\big|\; \underline{2a+\frac{3}{4a}-\frac{9}{64a^3}} \\
 & 4a^2 \\ \hline
4a+\frac{3}{4a} & 3 \\
 & 3+\frac{9}{16a^2} \\ \hline
4a+\frac{3}{2a}+\frac{9}{64a^3} & -\frac{9}{16a^2} \\
 & -\frac{9}{16a^2}-\frac{27}{128a^4}-\frac{81}{4096a^6} \\ \hline
\end{array}$$

33.

$$\begin{array}{r|l}
 & 4x^2-1 \;\big|\; \underline{2x-\frac{1}{4x}-\frac{1}{64x^3}} \\
 & 4x^2 \\ \hline
4x-\frac{1}{4x} & -1 \\
 & -1+\frac{1}{16x^2} \\ \hline
4x-\frac{1}{2x}-\frac{1}{64x^3} & -\frac{1}{16x^2} \\
 & -\frac{1}{16x^2}+\frac{1}{128x^4}+\frac{1}{4096x^6} \\ \hline
\end{array}$$

34.

$$\begin{array}{r|l}
1-2y^2 & 1-y^2-\frac{y^4}{2} \\ \cline{2-2}
1 & \\ \hline
2-y^2 & -2y^2 \\
 & -2y^2+y^4 \\ \hline
2-2y^2-\frac{y^4}{2} & -y^4 \\
 & -y^4+y^6+\frac{y^8}{4} \\ \hline
\end{array}$$

35.

$$\begin{array}{r|l}
a^2+\frac{1}{2}b & a+\frac{b}{4a}-\frac{b^2}{32a^3} \\ \cline{2-2}
a^2 & \\ \hline
2a+\frac{b}{4a} & \frac{1}{2}b \\
 & \frac{1}{2}b+\frac{b^2}{16a^2} \\ \hline
2a+\frac{b}{2a}-\frac{b^2}{32a^3} & -\frac{b^2}{16a^2} \\
 & -\frac{b^2}{16a^2}-\frac{b^3}{64a^4}+\frac{b^4}{1024a^6} \\ \hline
\end{array}$$

SQUARE ROOT OF NUMBERS.

Page 183.

30.

$$\begin{array}{r|l}
5.00\;00\;00\;00 & 2.2360+ \\ \cline{2-2}
4 & \\ \hline
40+2=42 & 100 \\
 & 84 \\ \hline
440+3=443 & 1600 \\
 & 1329 \\ \hline
4460+6=4466 & 27100 \\
 & 26796 \\ \hline
44720+0 & 30400 \\ \hline
\end{array}$$

31.

$$\begin{array}{r|l}
11.00\;00\;00\;00 & 3.3166+ \\ \cline{2-2}
9 & \\ \hline
60+3=63 & 200 \\
 & 189 \\ \hline
660+1=661 & 1100 \\
 & 661 \\ \hline
6620+6=6626 & 43900 \\
 & 39756 \\ \hline
66320+6=66326 & 414400 \\
 & 397956 \\ \hline
\end{array}$$

32.

$$\begin{array}{r|l}
13.00\;00\;00\;00 & 3.6055+ \\ \cline{2-2}
9 & \\ \hline
60+6=66 & 400 \\
 & 396 \\ \hline
7200+5=7205 & 40000 \\
 & 36025 \\ \hline
72100+5=72105 & 397500 \\
 & 360525 \\ \hline
\end{array}$$

33.

$$\begin{array}{r|l}
15.00\;00\;00\;00 & 3.8729+ \\ \cline{2-2}
9 & \\ \hline
60+8=68 & 600 \\
 & 544 \\ \hline
760+7=767 & 5600 \\
 & 5369 \\ \hline
7740+2=7742 & 23100 \\
 & 15484 \\ \hline
77440+9=77449 & 761600 \\
 & 697041 \\ \hline
\end{array}$$

34. 7.20 00 00 00 | 2.6832+
4
40+6=46 | 320
276
520+8=528 | 4400
4224
5360+3=5363 | 17600
16089
53660+2=53662 | 151100
107324

35. 5.30 00 00 00 | 2.3021+
4
40+3=43 | 130
129
4600+2=4602 | 10000
9204
46040+1=46041 | 79600
46041

36. .07 00 00 00 | .2645 +
4
40 + 6 = 46 | 300
276
520 + 4 = 524 | 2400
2096
5280 + 5 = 5285 | 30400
26425

37. .00 70 00 00 | .0836 +
64
160 + 3 = 163 | 600
489
1660 + 6 = 1666 | 11100
9996

38. $\frac{4}{7}$ = .57142857 +
.57 14 28 57 | .7559 +
49
140 + 5 = 145 | 814
725
1500 + 5 = 1505 | 8928
7525
15100 + 9 = 15109 | 140357
135981

39. $\frac{3}{8}$ = .375
.37 50 00 00 | .6123 +
36
120 + 1 = 121 | 150
121
1220 + 2 = 1222 | 2900
2444
12240 + 3 = 12243 | 45600
36729

40. $\frac{5}{9}$ = .55555555 +
.55 55 55 55 | .7453 +
49
140 + 4 = 144 | 655
576
1480 + 5 = 1485 | 7955
7425
14900 + 3 = 14903 | 53055
44709

41. $\frac{6}{11}$ = .54545454 +
.54 54 54 54 | .7385 +
49
140 + 3 = 143 | 554
429
1460 + 8 = 1468 | 12554
11744
14760 + 5 = 14765 | 81054
73825

42. $\frac{4}{3}$=1.33333333+
1.33 33 33 33 | 1.1547+
1
20+1=21 | 33
21
220+5=225 | 1233
1125
2300+4=2304 | 10833
9216
23080+7=23087 | 161733
161609

43. $\frac{3}{5}=.6$

$$\begin{array}{r|l}
\multicolumn{2}{r}{.60\ 00\ 00\ 00\ \underline{|\,.7745+}}\\
\multicolumn{2}{l}{\qquad\qquad\ \ 49}\\ \hline
140+7=147 & 1100\\
 & 1029\\ \hline
1540+4=1544 & 7100\\
 & 6176\\ \hline
15480+5=15485 & 92400\\
 & 77425\\ \hline
\end{array}$$

44. $\frac{7}{15}=.46\ 66\ 66\ 66+$

$$\begin{array}{r|l}
\multicolumn{2}{r}{.46\ 66\ 66\ 66\ \underline{|\,.6831+}}\\
\multicolumn{2}{l}{\qquad\qquad\ \ 36}\\ \hline
120+8=128 & 1066\\
 & 1024\\ \hline
1360+3=1363 & 4266\\
 & 4089\\ \hline
13660+1=13661 & 17766\\
 & 13661\\ \hline
\end{array}$$

45. $\frac{10}{15}=.66\ 66\ 66\ 66+$

$$\begin{array}{r|l}
\multicolumn{2}{r}{.66\ 66\ 66\ 66\ \underline{|\,.8164+}}\\
\multicolumn{2}{l}{\qquad\qquad\ \ 64}\\ \hline
160+1=161 & 266\\
 & 161\\ \hline
1620+6=1626 & 10566\\
 & 9756\\ \hline
16320+4=16324 & 81066\\
 & 65296\\ \hline
\end{array}$$

CUBE ROOT OF POLYNOMIALS.

Page 185.

3.

$$\begin{array}{r|l}
\multicolumn{2}{r}{x^3+6x^2y+12xy^2+8y^3\ \underline{|\,x+2y}}\\
\multicolumn{2}{l}{\qquad\qquad\qquad\quad x^3}\\ \hline
\text{Trial divisor, } 3x^2 & 6x^2y+12xy^2+8y^3\\
3x^2+6xy+4y^2 & 6x^2y+12xy^2+8y^3\\ \hline
\end{array}$$

4.

$$\begin{array}{r|l}
\multicolumn{2}{r}{27a^3+27a^2+9a+1\ \underline{|\,3a+1}}\\
\multicolumn{2}{l}{\qquad\qquad\qquad\quad 27a^3}\\ \hline
\text{Trial divisor, } 27a^2 & 27a^2+9a+1\\
27a^2+9a+1 & 27a^2+9a+1\\ \hline
\end{array}$$

5.

$$\begin{array}{r|l}
\multicolumn{2}{r}{8x^3-36x^2+54x-27\ \underline{|\,2x-3}}\\
\multicolumn{2}{l}{\qquad\qquad\qquad\quad 8x^3}\\ \hline
\text{Trial divisor, } 12x^2 & -36x^2+54x-27\\
12x^2-18x+9 & -36x^2+54x-27\\ \hline
\end{array}$$

6.

$$\begin{array}{r|l}
\multicolumn{2}{r}{27x^3+108x^2+144x+64\ \underline{|\,3x+4}}\\
\multicolumn{2}{l}{\qquad\qquad\qquad\quad 27x^3}\\ \hline
\text{Trial divisor, } 27x^2 & 108x^2+144x+64\\
27x^2+36x+16 & 108x^2+144x+64\\ \hline
\end{array}$$

Page 186.

7.

$$\begin{array}{r|l}
\multicolumn{1}{r}{a^3 + 3a + \frac{3}{a} + \frac{1}{a^3}} & \underline{a + \frac{1}{a}} \\
\multicolumn{1}{r}{a^3} & \\
\hline
\text{Trial divisor, } 3a^2 & 3a + \frac{3}{a} + \frac{1}{a^3} \\
3a^2 + 3 + \frac{1}{a^2} & 3a + \frac{3}{a} + \frac{1}{a^3} \\
\hline
\end{array}$$

8.

$$\begin{array}{r|l}
\multicolumn{1}{r}{a^3 - 12a^2 + 48a - 64} & \underline{a - 4} \\
\multicolumn{1}{r}{a^3} & \\
\hline
\text{Trial divisor, } 3a^2 & -12a^2 + 48a - 64 \\
3a^2 + 12a + 16 & -12a^2 + 48a - 64 \\
\hline
\end{array}$$

9.

$$\begin{array}{r|l}
\multicolumn{1}{r}{8a^3 + 12a^2 + 6a + 1} & \underline{2a + 1} \\
\multicolumn{1}{r}{8a^3} & \\
\hline
\text{Trial divisor, } 12a^2 & 12a^2 + 6a + 1 \\
12a^2 + 6a + 1 & 12a^2 + 6a + 1 \\
\hline
\end{array}$$

10.

$$\begin{array}{r|l}
\multicolumn{1}{r}{27x^6 - 54x^5 + 63x^4 - 44x^3 + 21x^2 - 6x + 1} & \underline{3x^2 - 2x + 1} \\
\multicolumn{1}{r}{27x^6} & \\
\hline
\text{Trial divisor, } 27x^4 & -54x^5 + 63x^4 - 44x^3 \\
27x^4 - 18x^3 + 4x^2 & -54x^5 + 36x^4 - 8x^3 \\
\hline
\text{Trial divisor, } 27x^4 & 27x^4 - 36x^3 + 21x^2 - 6x + 1 \\
27x^4 - 36x^3 + 21x^2 - 6x + 1 & 27x^4 - 36x^3 + 21x^2 - 6x + 1 \\
\hline
\end{array}$$

11.

$$\begin{array}{r|l}
\multicolumn{1}{r}{8m^6+36m^5+66m^4+63m^3+33m^2+9m+1} & \underline{2m^2+3m+1} \\
\multicolumn{1}{r}{8m^6} & \\
\hline
\text{Trial divisor, } 12m^4 & +36m^5+66m^4+63m^3 \\
12m^4+18m^3+9m^2 & +36m^5+54m^4+27m^3 \\
\hline
\text{Trial divisor, } 12m^4 & 12m^4+36m^3+33m^2+9m+1 \\
12m^4+36m^3+33m^2+9m+1 & 12m^4+36m^3+33m^2+9m+1 \\
\hline
\end{array}$$

12.

$$\begin{array}{r|l}
\multicolumn{1}{r}{1 - 3a + 6a^2 - 7a^3 + 6a^4 - 3a^5 + a^6} & \underline{1 - a + a^2} \\
\multicolumn{1}{r}{1} & \\
\hline
\text{Trial divisor, } 3 & -3a + 6a^2 - 7a^3 \\
3 - 3a + a^2 & -3a + 3a^2 - a^3 \\
\hline
\text{Trial divisor, } 3 & 3a^2 - 6a^3 + 6a^4 - 3a^5 + a^6 \\
3 - 6a + 6a^2 - 3a^3 + a^4 & 3a^2 - 6a^3 + 6a^4 - 3a^5 + a^6 \\
\hline
\end{array}$$

13.

$$\begin{array}{r|l}
\multicolumn{2}{r}{m^3-3m^2+5-\frac{3}{m^2}-\frac{1}{m^3}\ \Big|\ \underline{m-1-\frac{1}{m}}}\\
\multicolumn{2}{r}{\underline{m^3\qquad\qquad\qquad\qquad}\qquad\qquad\qquad}\\
\text{Trial divisor, } 3m^2 & -3m^2+5\\
3m^2-3m+1 & \underline{-3m^2+3m-1}\\
\text{Trial divisor, } 3m^2 & -3m+6-\frac{3}{m^2}-\frac{1}{m^3}\\
\underline{3m^2-6m+\frac{3}{m}+\frac{1}{m^2}} & \underline{-3m+6-\frac{3}{m^2}-\frac{1}{m^3}}
\end{array}$$

14.

$$\begin{array}{r|l}
\multicolumn{2}{r}{x^3-3x^2y+3xy^2-y^3+6x^2z-12xyz+6y^2z+12xz^2-12yz^2+8z^3\ \big|\ x-y+2z}\\
\multicolumn{2}{l}{\underline{x^3}}\\
\text{Trial divisor, } 3x^2 & -3x^2y+3xy^2-y^3\\
3x^2-3xy+y^2 & \underline{-3x^2y+3xy^2-y^3}\\
\text{Trial divisor, } 3x^2 & 6x^2z-12xyz+6y^2z+12xz^2-12yz^2+8z^3\\
\underline{3x^3-6xy+3y^2+6xz-6yz+4z^3} & \underline{6x^2z-12xyz+6y^2z+12xz^2-12yz^2+8z^3}
\end{array}$$

15.

$$\begin{array}{r|l}
\multicolumn{2}{r}{8a^3-84a^2b+294ab^2-343b^3\ \big|\ \underline{2a-7b}}\\
\multicolumn{2}{r}{\underline{8a^3\qquad\qquad\qquad\qquad\qquad}}\\
\text{Trial divisor, } 12a^2 & -84a^2b+294ab^2-343b^3\\
\underline{12a^2+42ab-49b^2} & \underline{-84a^2b+294ab^2-343b^3}
\end{array}$$

16.

$$\begin{array}{r|l}
\multicolumn{2}{r}{y^6-6y^5+21y^4-44y^3+63y^2-54y+27\ \big|\ \underline{y^2-2y+3}}\\
\multicolumn{2}{r}{\underline{y^6\qquad\qquad\qquad\qquad\qquad}}\\
\text{Trial divisor, } 3y^4 & -6y^5+21y^4-44y^3\\
3y^4-6y^3+4y^4 & \underline{-6y^5+12y^4-8y^3}\\
\text{Trial divisor, } 3y^4-12y^3+12y^2 & 9y^4-36y^3+63y^2-54y+27\\
\underline{3y^4-12y^3+21y^2-18y+9} & \underline{9y^4-36y^3+63y^2-54y+27}
\end{array}$$

17.

$$\begin{array}{r|l}
\multicolumn{2}{r}{1-9a+39a^2-99a^3+156a^4-144a^5+64a^6\ \big|\ \underline{1-3a+4a^2}}\\
\multicolumn{2}{r}{\underline{1\qquad\qquad\qquad\qquad\qquad}}\\
\text{Trial divisor, } 3 & -9a+39a^2-99a^3\\
3-9a+9a^2 & \underline{-9a+27a^2-27a^3}\\
\text{Trial divisor, } 3-18a+27a^2 & 12a^2-72a^3+156a^4-144a^5+64a^6\\
\underline{3-18a+39a^2-36a^3+16a^4} & \underline{12a^2-72a^3+156a^4-144a^5+64a^6}
\end{array}$$

18.

$$\begin{array}{r|l}
\multicolumn{2}{r}{8c^6-36c^5+66c^4-63c^3+33c^2-9c+1\ \big|\ \underline{2c^2-3c+1}}\\
\multicolumn{2}{r}{\underline{8c^6\qquad\qquad\qquad\qquad\qquad}}\\
\text{Trial divisor, } 12c^4 & -36c^5+66c^4-63c^3\\
12c^4-18c^3+9c^2 & \underline{-36c^5+54c^4-27c^3}\\
\text{Trial divisor, } 12c^4-36c^3+27c^2 & 12c^4-36c^3+33c^2-9c+1\\
\underline{12c^4-36c^3+33c^2-9c+1} & \underline{12c^4-36c^3+33c^2-9c+1}
\end{array}$$

19.

$$a^6+3a^4+6a^2+7+\frac{6}{a^2}+\frac{3}{a^4}+\frac{1}{a^6}\;\Big|\;\underline{a^2+1+\frac{1}{a^2}}$$

$$\begin{array}{r|l}
 & \underline{a^6} \\
\text{Trial divisor, } 3a^4 & 3a^4+6a^2+7 \\
3a^4+3a^2+1 & 3a^4+3a^2+1 \\ \hline
\text{Trial divisor, } 3a^4+6a^2+3 & 3a^2+6+\frac{6}{a^2}+\frac{3}{a^4}+\frac{1}{a^6} \\
3a^4+6a^2+6+\frac{3}{a^2}+\frac{1}{a^4} & 3a^2+6+\frac{6}{a^2}+\frac{3}{a^4}+\frac{1}{a^6} \\ \hline
\end{array}$$

20.

$$x^6-12x^5y+60x^4y^2-160x^3y^3+240x^2y^4-192xy^5+64y^6\;\Big|\;\underline{x^2-4xy+4y^2}$$

$$\begin{array}{r|l}
 & \underline{x^6} \\
\text{Trial divisor, } 3x^4 & -12x^5y+60x^4y^2-160x^3y^3 \\
3x^4-12x^3y+16x^2y^2 & -12x^5y+48x^4y^2-64x^3y^3 \\ \hline
\text{Trial divisor, } 3x^4-24x^3y+48x^2y^2 & 12x^4y^2-96x^3y^3+240x^2y^4-192xy^5+64y^6 \\
3x^4-24x^3y+60x^2y^2-48xy^3+16y^4 & 12x^4y^2-96x^3y^3+240x^2y^4-192xy^5+64y^6 \\ \hline
\end{array}$$

ANY ROOTS OF POLYNOMIALS.

Page 187.

1. To find the fourth root, we may extract the square root, and then the square root of that root.

The square root of $1-8a+24a^2-32a^3+16a^4$ is $1-4a+4a^2$; and the square root of $1-4a+4a^2$ is $1-2a$, which is, therefore, the fourth root of the given quantity.

2.

$$x^5+5x^4+10x^3+10x^2+5x+1\;\Big|\;\underline{x+1}$$

$$\begin{array}{r|l}
 & \underline{x^5} \\
\text{Trial divisor, } 5x^4 & 5x^4+10x^3+10x^2+5x+1 \\
5x^4+10x^3+10x^2+5x+1 & 5x^4+10x^3+10x^2+5x+1 \\ \hline
\end{array}$$

3. By extracting the square root of the cube root the sixth root is obtained.

The cube root of $x^6+6x^5+15x^4+20x^3+15x^2+6x+1$ is x^2+2x+1; and the square root of this result is $x+1$, which is the sixth root of the given quantity.

4. The fourth root equals the square root of the square root.

The square root of $a^4+8a^3b+24a^2b^2+32ab^3+16b^4$ is $a^2+4ab+4b^2$; and the square root of $a^2+4ab+4b^2$ is $a+2b$, which is, therefore, the fourth root of the given quantity.

5.

$$\begin{array}{r|l}
 & x^5-5x^4y+10x^3y^2-10x^2y^3+5xy^4-y^5 \;\underline{|\,x-y} \\
 & x^5 \\ \hline
\text{Trial divisor, } 5x^4 & -5x^4y+10x^3y^2-10x^2y^3+5xy^4-y^5 \\
5x^4-10x^3y+10x^2y^2-5xy^3+y^4 & -5x^4y+10x^3y^2-10x^2y^3+5xy^4-y^5 \\ \hline
\end{array}$$

6.

$$\begin{array}{r|l}
 & a^5-\frac{5a^4x}{y}+\frac{10a^3x^2}{y^2}-\frac{10a^2x^3}{y^3}+\frac{5ax^4}{y^4}-\frac{x^5}{y^5} \;\underline{|\,a-\frac{x}{y}} \\
 & a^5 \\ \hline
\text{Trial divisor, } 5a^4 & -\frac{5a^4x}{y}+\frac{10a^3x^2}{y^2}-\frac{10a^2x^3}{y^3}+\frac{5ax^4}{y^4}-\frac{x^5}{y^5} \\
5a^4-\frac{10a^3x}{y}+\frac{10a^2x^2}{y^2}-\frac{5ax^3}{y^3}+\frac{x^4}{y^4} & -\frac{5a^4x}{y}+\frac{10a^3x^2}{y^2}-\frac{10a^2x^3}{y^3}+\frac{5ax^4}{y^4}-\frac{x^5}{y^5} \\ \hline
\end{array}$$

7. The square root of the quantity is $c^3 + 3c^2z + 3cz^2 + z^3$; and the cube root of this is $c + z$, which is, therefore, the sixth root of the quantity.

CUBE ROOT OF NUMBERS.

Page 190.

18.

$$\begin{array}{r|l}
 & 3.000\,000\,000 \;\underline{|\,1.442+} \\
 & 1 \\ \hline
300 & 2000 \\
136 & \\
436 & 1744 \\ \hline
58800 & 256000 \\
1696 & \\
60496 & 241984 \\ \hline
6220800 & 14016000 \\
8644 & \\
6229444 & 12458888 \\ \hline
\end{array}$$

19.

$$\begin{array}{r|l}
 & .270\,000\,000 \;\underline{|\,.646+} \\
 & 216 \\ \hline
10800 & 54000 \\
736 & \\
11536 & 46144 \\ \hline
1228800 & 7856000 \\
11556 & \\
1240356 & 7442136 \\ \hline
\end{array}$$

20.

$$\begin{array}{r|l}
 & 6.400\,000\,000 \;\underline{|\,1.856+} \\
 & 1 \\ \hline
300 & 5400 \\
304 & \\
604 & 4832 \\ \hline
97200 & 568000 \\
2725 & \\
99925 & 499625 \\ \hline
10267500 & 68375000 \\
33336 & \\
10300836 & 61805016 \\ \hline
\end{array}$$

21.

$$\begin{array}{r|l}
 & .004\,650\,000 \;\underline{|\,.166+} \\
 & 1 \\ \hline
300 & 3650 \\
216 & \\
516 & 3096 \\ \hline
76800 & 554000 \\
2916 & \\
79716 & 478296 \\ \hline
\end{array}$$

22. $\frac{5}{6} = .833\ 333\ 333+$

$$\begin{array}{r|l}
\multicolumn{2}{r}{.833\ 333\ 333\ \lfloor\underline{.941+}}\\
\multicolumn{2}{c}{729}\\ \hline
24300 & 104333\\
1096 & \\
25396 & 101584\\ \hline
2650800 & 2749333\\
2821 & \\
2653621 & 2653621\\ \hline
\end{array}$$

23. $\frac{7}{8} = .875$

$$\begin{array}{r|l}
\multicolumn{2}{r}{.875\ 000\ 000\ \lfloor\underline{.956+}}\\
\multicolumn{2}{c}{729}\\ \hline
24300 & 146000\\
1375 & \\
25675 & 128375\\ \hline
2705000 & 17625000\\
17136 & \\
2722136 & 16332816\\ \hline
\end{array}$$

24. $\frac{3}{11} = .272\ 727\ 272+$

$$\begin{array}{r|l}
\multicolumn{2}{r}{.272\ 727\ 272+\ \lfloor\underline{.648+}}\\
\multicolumn{2}{c}{216}\\ \hline
10800 & 56727\\
736 & \\
11536 & 46144\\ \hline
1228800 & 10583272\\
15424 & \\
1244224 & 9953792\\ \hline
\end{array}$$

25. $\frac{4}{15} = .266666666+$

$$\begin{array}{r|l}
\multicolumn{2}{r}{.266\ 666\ 666+\ \lfloor\underline{.643+}}\\
\multicolumn{2}{c}{216}\\ \hline
10800 & 50666\\
736 & \\
11536 & 46144\\ \hline
1228800 & 4522666\\
5769 & \\
1234569 & 3703707\\ \hline
\end{array}$$

THEORY OF EXPONENTS.

Page 192.

17. $16^{\frac{3}{2}} = (\sqrt{16})^3 = 4^3 = 64.$

18. $25^{\frac{1}{2}} = \sqrt{25} = 5.$

19. $(-27)^{\frac{4}{3}} = (\sqrt[3]{-27})^4 = (-3)^4 = 81.$

20. $-27^{\frac{4}{3}} = -(\sqrt[3]{27})^4 = -3^4 = -81.$

21. $36^{\frac{3}{2}} = (\sqrt{36})^3 = 6^3 = 216.$

22. $9^{\frac{5}{2}} = (\sqrt{9})^5 = 3^5 = 243.$

23. $(\frac{9}{16})^{\frac{3}{2}} = (\sqrt{\frac{9}{16}})^3 = (\frac{3}{4})^3 = \frac{27}{64}.$

24. $(\frac{81}{256})^{\frac{3}{4}} = (\sqrt[4]{\frac{81}{256}})^3 = (\frac{3}{4})^3 = \frac{27}{64}.$

Page 195.

24.

$$\begin{array}{l}
2x^{\frac{1}{3}} + x + 3x^{\frac{2}{3}}\\
\underline{\quad x^{\frac{2}{3}} + 2\qquad\qquad}\\
2x + x^{\frac{5}{3}} + 3x^{\frac{4}{3}} + 4x^{\frac{1}{3}} + 6x^{\frac{2}{3}}\\
\underline{2x\qquad\qquad\qquad\qquad\qquad\qquad\quad}\\
4x + x^{\frac{5}{3}} + 3x^{\frac{4}{3}} + 4x^{\frac{1}{3}} + 6x^{\frac{2}{3}}
\end{array}$$

25.

$$\begin{array}{llllll}
x^{\frac{1}{4}} & + x^{\frac{1}{2}}y^{\frac{1}{2}} & + y^{\frac{1}{4}} & & & \\
x^{\frac{1}{4}} & - y^{\frac{1}{4}} & & & & \\ \hline
x^{\frac{1}{2}} & + x^{\frac{3}{4}}y^{\frac{1}{2}} & + x^{\frac{1}{4}}y^{\frac{1}{4}} & & & \\
 & & - x^{\frac{1}{4}}y^{\frac{1}{4}} & - x^{\frac{1}{2}}y^{\frac{3}{4}} & - y^{\frac{1}{2}} & \\ \hline
x^{\frac{1}{2}} & + x^{\frac{3}{4}}y^{\frac{1}{2}} & & - x^{\frac{1}{2}}y^{\frac{3}{4}} & - y^{\frac{1}{2}} &
\end{array}$$

26.
$$\begin{array}{r}
a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}} \\
a^{-\frac{1}{3}} - b^{-\frac{1}{3}} \\
\hline
a^{\frac{1}{3}} + b^{\frac{1}{3}} + a^{-\frac{1}{3}}b^{\frac{2}{3}} - a^{\frac{2}{3}}b^{-\frac{1}{3}} \\
-a^{\frac{1}{3}} - b^{\frac{1}{3}} \\
\hline
a^{-\frac{1}{3}}b^{\frac{2}{3}} - a^{\frac{2}{3}}b^{-\frac{1}{3}}
\end{array}$$

27.
$$\begin{array}{l}
x^2 + x + 1 \\
x - x^{-1} - 1 \\
\hline
x^3 + x^2 + x \\
\qquad\quad - x - 1 - x^{-1} \\
\quad - x^2 - x - 1 \\
\hline
x^3 \qquad - x - 2 - x^{-1}
\end{array}$$

28.
$$\begin{array}{l}
x^3y^{-3} - 3 + x^{-3}y^3 \\
x^3y^{-3} + 3 + x^{-3}y^3 \\
\hline
x^6y^{-6} - 3x^3y^{-3} + 1 \\
\qquad\qquad 3x^3y^{-3} - 9 + 3x^{-3}y^3 \\
\qquad\qquad\qquad\qquad 1 - 3x^{-3}y^3 + x^{-6}y^6 \\
\hline
x^6y^{-6} - \qquad\qquad 7 \qquad\qquad + x^{-6}y^6
\end{array}$$

29.
$$\begin{array}{l}
x^2y^{-2} - xy + x^{-2}y^2 \\
x + xy - y \\
\hline
x^3y^{-2} - x^2y + x^{-1}y^2 + x^3y^{-1} - x^2y^2 + x^{-1}y^3 - x^2y^{-1} + xy^2 - x^{-2}y^3
\end{array}$$

30.
$$\begin{array}{l}
x^{-3} - y^{-3} + x^{-\frac{3}{2}}y^{-\frac{3}{2}} \\
x^3 + y^3 \\
\hline
1 - x^3y^{-3} + x^{\frac{3}{2}}y^{-\frac{3}{2}} + x^{-3}y^3 + x^{-\frac{3}{2}}y^{\frac{3}{2}} \\
-1 \\
\hline
\quad - x^3y^{-3} + x^{\frac{3}{2}}y^{-\frac{3}{2}} + x^{-3}y^3 + x^{-\frac{3}{2}}y^{\frac{3}{2}}
\end{array}$$

Page 196.

19.
$$\begin{array}{l|l}
x - y & x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}} \\
\cline{2-2}
x + x^{\frac{2}{3}}y^{\frac{1}{3}} + x^{\frac{1}{3}}y^{\frac{2}{3}} & x^{\frac{1}{3}} - \quad y^{\frac{1}{3}} \\
\hline
\quad - x^{\frac{2}{3}}y^{\frac{1}{3}} - x^{\frac{1}{3}}y^{\frac{2}{3}} - y & \\
\quad - x^{\frac{2}{3}}y^{\frac{1}{3}} - x^{\frac{1}{3}}y^{\frac{2}{3}} - y & \\
\cline{1-1}
\end{array}$$

Page 197.

20.
$$\begin{array}{l|l}
x^{-3} + x^{-2}y^{-1} + x^{-1}y^{-2} + y^{-3} & x^{-1} + y^{-1} \\
\cline{2-2}
x^{-3} + x^{-2}y^{-1} & x^{-2} + y^{-2} \\
\hline
\qquad\qquad x^{-1}y^{-2} + y^{-3} & \\
\qquad\qquad x^{-1}y^{-2} + y^{-3} & \\
\cline{1-1}
\end{array}$$

21.

$$\begin{array}{l|l}
x^{-6}-3x^{-4}y^{-2}+3x^{-2}y^{-4}-y^{-6} & x^{-2}-y^{-2} \\ \cline{2-2}
x^{-6}-x^{-4}y^{-2} & x^{-4}-2x^{-2}y^{-2}+y^{-4} \\ \hline
\multicolumn{2}{l}{\quad -2x^{-4}y^{-2}+3x^{-2}y^{-4}} \\
\multicolumn{2}{l}{\quad \underline{-2x^{-4}y^{-2}+2x^{-2}y^{-4}}} \\
\multicolumn{2}{l}{\qquad\qquad\quad x^{-2}y^{-4}-y^{-6}} \\
\multicolumn{2}{l}{\qquad\qquad\quad \underline{x^{-2}y^{-4}-y^{-6}}}
\end{array}$$

22.

$$\begin{array}{r|l}
a^{-2}b^{2}+a^{-1}b+ab^{-1}-a^{2}b^{-2} & a^{2}+b^{2} \\ \cline{2-2}
-1-a^{2}b^{-2} & -b^{-2}+a^{-1}b^{-1}+a^{-2} \\ \hline
\multicolumn{2}{l}{\qquad a^{-1}b+ab^{-1}+1} \\
\multicolumn{2}{l}{\qquad \underline{a^{-1}b+ab^{-1}}} \\
\multicolumn{2}{l}{\qquad\qquad\quad a^{-2}b^{2}+1} \\
\multicolumn{2}{l}{\qquad\qquad\quad \underline{a^{-2}b^{2}+1}}
\end{array}$$

23.

$$\begin{array}{l|l}
c^{\frac{4}{3}}+c+c^{\frac{1}{3}}+1 & c^{\frac{2}{3}}+2c^{\frac{1}{3}}+1 \\ \cline{2-2}
c^{\frac{4}{3}}+2c+c^{\frac{2}{3}} & c^{\frac{2}{3}}-c^{\frac{1}{3}}+1 \\ \hline
\multicolumn{2}{l}{\quad -c-c^{\frac{2}{3}}+c^{\frac{1}{3}}+1} \\
\multicolumn{2}{l}{\quad \underline{-c-2c^{\frac{2}{3}}-c^{\frac{1}{3}}}} \\
\multicolumn{2}{l}{\qquad\qquad c^{\frac{2}{3}}+2c^{\frac{1}{3}}+1} \\
\multicolumn{2}{l}{\qquad\qquad \underline{c^{\frac{2}{3}}+2c^{\frac{1}{3}}+1}}
\end{array}$$

Page 198.

15.

$$\begin{array}{r|l}
\multicolumn{2}{r}{9c-4c^{-\frac{1}{2}}+10-12c^{\frac{1}{2}}+c^{-1}\ \big|\ \underline{3c^{\frac{1}{2}}-2+c^{-\frac{1}{2}}}} \\
\multicolumn{2}{l}{\qquad 9c} \\ \hline
6c^{\frac{1}{2}}-2 & -4c^{-\frac{1}{2}}+10-12c^{\frac{1}{2}}+c^{-1} \\
 & \qquad +4-12c^{-\frac{1}{2}} \\ \hline
6c^{\frac{1}{2}}-4+c^{-\frac{1}{2}} & 6-4c^{-\frac{1}{2}}+c^{-1} \\
 & \underline{6-4c^{-\frac{1}{2}}+c^{-1}}
\end{array}$$

16

$$\begin{array}{r|l}
\multicolumn{2}{r}{4a+12a^{\frac{1}{2}}b^{\frac{1}{3}}+4a^{\frac{1}{2}}c+9b^{\frac{2}{3}}+6b^{\frac{1}{3}}c+c^{2}\ \big|\ \underline{2a^{\frac{1}{2}}+3b^{\frac{1}{3}}+c}} \\
\multicolumn{2}{l}{\qquad 4a} \\ \hline
4a^{\frac{1}{2}}+3b^{\frac{1}{3}} & 12a^{\frac{1}{2}}b^{\frac{1}{3}}+4a^{\frac{1}{2}}c+9b^{\frac{2}{3}} \\
 & 12a^{\frac{1}{2}}b^{\frac{1}{3}}\qquad +9b^{\frac{2}{3}} \\ \hline
4a^{\frac{1}{2}}+6b^{\frac{1}{3}}+c & 4a^{\frac{1}{2}}c\qquad +6b^{\frac{1}{3}}c+c^{2} \\
 & \underline{4a^{\frac{1}{2}}c\qquad +6b^{\frac{1}{3}}c+c^{2}}
\end{array}$$

17.

$$\begin{array}{r|l}
\multicolumn{2}{l}{x^{-\frac{2}{3}} + 4x^{\frac{1}{3}} - 10x^{-\frac{1}{3}} + 4x^{\frac{4}{3}} - 20x^{\frac{2}{3}} + 25 \;\underline{\big|\, x^{-\frac{1}{3}} - 5 + 2x^{\frac{2}{3}}}} \\
\multicolumn{2}{l}{x^{-\frac{2}{3}}} \\ \hline
2x^{-\frac{1}{3}} - 5 & -10x^{-\frac{1}{3}} + 25 \\
 & -10x^{-\frac{1}{3}} + 25 \\ \hline
2x^{-\frac{1}{3}} - 10 + 2x^{\frac{2}{3}} & 4x^{\frac{1}{3}} + 4x^{\frac{4}{3}} - 20x^{\frac{2}{3}} \\
 & \underline{4x^{\frac{1}{3}} + 4x^{\frac{4}{3}} - 20x^{\frac{2}{3}}}
\end{array}$$

18.

$$\begin{array}{r|l}
\multicolumn{2}{l}{x + 2x^{\frac{3}{4}} + 3x^{\frac{1}{2}} + 2x^{\frac{1}{4}} + 1 \;\underline{\big|\, x^{\frac{1}{2}} + x^{\frac{1}{4}} + 1}} \\
\multicolumn{2}{l}{x} \\ \hline
2x^{\frac{1}{2}} + x^{\frac{1}{4}} & 2x^{\frac{3}{4}} + 3x^{\frac{1}{2}} \\
 & 2x^{\frac{3}{4}} + x^{\frac{1}{2}} \\ \hline
2x^{\frac{1}{2}} + 2x^{\frac{1}{4}} + 1 & 2x^{\frac{1}{2}} + 2x^{\frac{1}{4}} + 1 \\
 & \underline{2x^{\frac{1}{2}} + 2x^{\frac{1}{4}} + 1}
\end{array}$$

19.

$$\begin{array}{r|l}
\multicolumn{2}{l}{4x^{\frac{1}{2}} - 20x^{\frac{3}{8}}y^{\frac{1}{8}} + 37x^{\frac{1}{4}}y^{\frac{1}{4}} - 30x^{\frac{1}{8}}y^{\frac{3}{8}} + 9y^{\frac{1}{2}} \;\underline{\big|\, 2x^{\frac{1}{4}} - 5x^{\frac{1}{8}}y^{\frac{1}{8}} + 3y^{\frac{1}{4}}}} \\
\multicolumn{2}{l}{4x^{\frac{1}{2}}} \\ \hline
4x^{\frac{1}{4}} - 5x^{\frac{1}{8}}y^{\frac{1}{8}} & -20x^{\frac{3}{8}}y^{\frac{1}{8}} + 37x^{\frac{1}{4}}y^{\frac{1}{4}} \\
 & -20x^{\frac{3}{8}}y^{\frac{1}{8}} + 25x^{\frac{1}{4}}y^{\frac{1}{4}} \\ \hline
4x^{\frac{1}{4}} - 10x^{\frac{1}{8}}y^{\frac{1}{8}} + 3y^{\frac{1}{4}} & 12x^{\frac{1}{4}}y^{\frac{1}{4}} - 30x^{\frac{1}{8}}y^{\frac{3}{8}} + 9y^{\frac{1}{2}} \\
 & \underline{12x^{\frac{1}{4}}y^{\frac{1}{4}} - 30x^{\frac{1}{8}}y^{\frac{3}{8}} + 9y^{\frac{1}{2}}}
\end{array}$$

20.

$$\begin{array}{r|l}
\multicolumn{2}{l}{x + 12x^{\frac{2}{3}} + 48x^{\frac{1}{3}} + 64 \;\underline{\big|\, x^{\frac{1}{3}} + 4}} \\
\multicolumn{2}{l}{x} \\ \hline
3x^{\frac{2}{3}} & 12x^{\frac{2}{3}} + 48x^{\frac{1}{3}} + 64 \\
3x^{\frac{2}{3}} + 12x^{\frac{1}{3}} + 16 & \underline{12x^{\frac{2}{3}} + 48x^{\frac{1}{3}} + 64}
\end{array}$$

21.

$$\begin{array}{r|l}
\multicolumn{2}{l}{x^2 - 3x^{\frac{5}{3}} + 5x - 3x^{\frac{1}{3}} - 1 \;\underline{\big|\, x^{\frac{2}{3}} - x^{\frac{1}{3}} - 1}} \\
\multicolumn{2}{l}{x^2} \\ \hline
3x^{\frac{4}{3}} & -3x^{\frac{5}{3}} \qquad\qquad + 5x \\
3x^{\frac{4}{3}} - 3x + x^{\frac{2}{3}} & -3x^{\frac{5}{3}} + 3x^{\frac{4}{3}} - x \\ \hline
3x^{\frac{4}{3}} - 6x + 3x^{\frac{2}{3}} & -3x^{\frac{4}{3}} + 6x - 3x^{\frac{1}{3}} - 1 \\
3x^{\frac{4}{3}} - 6x + 3x^{\frac{1}{3}} + 1 & \underline{-3x^{\frac{4}{3}} + 6x - 3x^{\frac{1}{3}} - 1}
\end{array}$$

22.

$$\begin{array}{r|l}
 & \frac{1}{8}x^3 - \frac{3}{2}x^2y^{\frac{1}{2}} + 6xy - 8y^{\frac{3}{2}} \;\underline{\big|\,\frac{1}{2}x - 2y^{\frac{1}{2}}} \\
 & \underline{\frac{1}{8}x^3 \qquad\qquad\qquad\qquad\qquad} \\
\frac{3}{4}x^2 & -\frac{3}{2}x^2y^{\frac{1}{2}} + 6xy - 8y^{\frac{3}{2}} \\
\underline{\frac{3}{4}x^2 - 3xy^{\frac{1}{2}} + 4y} & \underline{-\frac{3}{2}x^2y^{\frac{1}{2}} + 6xy - 8y^{\frac{3}{2}}}
\end{array}$$

23.

$$\begin{array}{r|l}
 & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{\big|\,2a+1-3a^{-1}} \\
 & 8a^3+12a^2-30a-35+45a^{-1}+27a^{-2}-27a^{-3} \\
 & \underline{8a^2 \qquad\qquad\qquad} \\
12a^2 & 12a^2-30a-35 \\
12a^2+6a+1 & \underline{12a^2+\ 6a+\ 1 \qquad\qquad\qquad\qquad\qquad} \\
12a^2+12a+3 & -36a-36+45a^{-1}+27a^{-2}-27a^{-3} \\
\underline{12a^2+12a-15-9a^{-1}+9a^{-2}} & \underline{-36a-36+45a^{-1}+27a^{-2}-27a^{-3}}
\end{array}$$

REDUCTION OF RADICALS.

Page 201.

32. $\sqrt[3]{(a-x)^5y^4} = \sqrt[3]{y^3(a-x)^3(a-x)^2y} = y(a-x)\sqrt[3]{(a-x)^2y}.$

33. $a\sqrt[3]{a^4-a^3x^2} = a\sqrt[3]{a^3(a-x^2)} = a^2\sqrt[3]{a-x^2}.$

34. $a\sqrt{a^3+2a^2b+ab^2} = a\sqrt{(a^2+2ab+b^2)a} = a(a+b)\sqrt{a}.$

35. $(a+b)\sqrt{a^4-a^2b^2} = (a+b)\sqrt{a^2(a^2-b^2)} = a(a+b)\sqrt{a^2-b^2}.$

36. $(x+y)\sqrt{x^3-2x^2y+xy^2} = (x+y)\sqrt{(x^2-2xy+y^2)x} = (x^2-y^2)\sqrt{x}.$

37. $(x^2+x^2y+x^2y^2)^{\frac{1}{2}} = \{x^2(1+y+y^2)\}^{\frac{1}{2}} = x(1+y+y^2)^{\frac{1}{2}}.$

38. $4a(x+2xy+xy^2)^{\frac{1}{2}} = 4a\{x(1+2y+y^2)\}^{\frac{1}{2}} = 4a(1+y)x^{\frac{1}{2}}.$

Page 202.

40. $\sqrt{\frac{2}{3}} = \sqrt{\frac{2\times 3}{3\times 3}} = \sqrt{\frac{6}{9}} = \sqrt{\frac{1}{9}\times 6} = \frac{1}{3}\sqrt{6}.$

41. $\sqrt{\frac{3}{7}} = \sqrt{\frac{3\times 7}{7\times 7}} = \sqrt{\frac{21}{49}} = \sqrt{\frac{1}{49}\times 21} = \frac{1}{7}\sqrt{21}.$

42. $\sqrt{\frac{5}{8}} = \sqrt{\frac{5\times 8}{8\times 8}} = \sqrt{\frac{40}{64}} = \sqrt{\frac{4}{64}\times 40} = \frac{1}{4}\sqrt{10}.$

$\frac{1}{4}\sqrt{10}$

43. $\sqrt[3]{\frac{5}{9}}=\sqrt[3]{\frac{5\times 3}{9\times 3}}=\sqrt[3]{\frac{15}{27}}=\sqrt[3]{\frac{1}{27}\times 15}=\frac{1}{3}\sqrt[3]{15}.$

44. $\sqrt{\frac{5a}{7b}}=\sqrt{\frac{5a\times 7b}{7b\times 7b}}=\sqrt{\frac{35ab}{49b^2}}=\sqrt{\frac{1}{49b^2}}\times\sqrt{35ab}=\frac{1}{7b}\sqrt{35ab}.$

45. $\sqrt{\frac{7x}{9y}}=\sqrt{\frac{7x\times y}{9y\times y}}=\sqrt{\frac{7xy}{9y^2}}=\sqrt{7xy\times\frac{1}{9y^2}}=\frac{1}{3y}\sqrt{7xy}.$

46. $\sqrt{\frac{5xy}{6ab}}=\sqrt{\frac{5xy\times 6ab}{6ab\times 6ab}}=\sqrt{\frac{30abxy}{36a^2b^2}}=\sqrt{\frac{1}{36a^2b^2}\times 30abxy}$

$=\frac{1}{6ab}\sqrt{30abxy}.$

47. $\sqrt[3]{\frac{7a}{8b}}=\sqrt[3]{\frac{7a\times b^2}{8b\times b^2}}=\sqrt[3]{\frac{7ab^2}{8b^3}}=\sqrt[3]{\frac{1}{8b^3}\times 7ab^2}=\frac{1}{2b}\sqrt[3]{7ab^2}.$

48. $\sqrt[3]{\frac{2c}{3x}}=\sqrt[3]{\frac{2c\times 9x^2}{3x\times 9x^2}}=\sqrt[3]{\frac{18cx^2}{27x^3}}=\sqrt[3]{\frac{1}{27x^3}\times 18cx^2}=\frac{1}{3x}\sqrt[3]{18cx^2}.$

49. $\sqrt{\frac{3x^3}{5a}}=\sqrt{\frac{3x^3\times 5a}{5a\times 5a}}=\sqrt{\frac{15ax^3}{25a^2}}=\sqrt{\frac{x^2}{25a^2}\times 15ax}=\frac{x}{5a}\sqrt{15ax}.$

50. $2\sqrt{\frac{5}{8}}=2\sqrt{\frac{5\times 2}{8\times 2}}=2\sqrt{\frac{10}{16}}=2\sqrt{\frac{1}{16}\times 10}=\frac{1}{2}\sqrt{10}.$

51. $3\sqrt{\frac{7a}{12b}}=3\sqrt{\frac{7a\times 3b}{12b\times 3b}}=3\sqrt{\frac{21ab}{36b^2}}=3\sqrt{21ab\times\frac{1}{36b^2}}=\frac{1}{2b}\sqrt{21ab}.$

52. $(x-2)\sqrt{\frac{x-2}{x+2}}=(x-2)\sqrt{\frac{x-2\times(x+2)}{x+2\times(x+2)}}$

$=(x-2)\sqrt{(x^2-4)\times\frac{1}{(x+2)^2}}=\frac{x-2}{x+2}\sqrt{x^2-4}.$

53. $(a+b)\sqrt{\frac{a}{a-b}}=(a+b)\sqrt{\frac{a(a-b)}{(a-b)(a-b)}}$

$=(a+b)\sqrt{\frac{1}{(a-b)^2}}\times\sqrt{a(a-b)}=\frac{a+b}{a-b}\sqrt{a(a-b)}.$

54. $(x+y)\sqrt[3]{\frac{x}{(x+y)^2}} = (x+y)\sqrt[3]{\frac{x(x+y)}{(x+y)^2(x+y)}}$

$= (x+y)\sqrt[3]{\frac{1}{(x+y)^3}} \times \sqrt[3]{x(x+y)} = \frac{x+y}{x+y}\sqrt[3]{x(x+y)} = \sqrt[3]{x(x+y)}.$

55. $(a^2-b^2)\sqrt{\frac{x}{a^2+b^2}} = (a^2-b^2)\sqrt{\frac{x(a^2+b^2)}{(a^2+b^2)(a^2+b^2)}}$

$= (a^2-b^2)\sqrt{\frac{1}{(a^2+b^2)^2}} \times \sqrt{x(a^2+b^2)} = \frac{a^2-b^2}{a^2+b^2}\sqrt{x(a^2+b^2)}.$

56. $3a\left(\frac{3x}{7y}\right)^{\frac{1}{2}} = 3a\left(\frac{3x\cdot 7y}{49y^2}\right)^{\frac{1}{2}} = 3a\left(\frac{1}{49y^2}\right)^{\frac{1}{2}} \times (21xy)^{\frac{1}{2}} = \frac{3a}{7y}(21xy)^{\frac{1}{2}}.$

57. $5xy\left(\frac{3x^2y}{4x^2y^2}\right)^{\frac{1}{3}} = 5xy\left(\frac{3x^2y \times 2xy}{8x^3y^3}\right)^{\frac{1}{3}} = 5xy\left(\frac{x^3}{8x^3y^3}\right)^{\frac{1}{3}} \times (6y^2)^{\frac{1}{3}}$

$= \frac{5x}{2}(6y^2)^{\frac{1}{3}}.$

Page 203.

28. $(x^2-y^2)\sqrt{\frac{x}{(x-y)^2}} = \sqrt{\frac{x(x^2-y^2)^2}{(x-y)^2}} = \sqrt{\frac{x(x+y)^2(x-y)^2}{(x-y)^2}} = \sqrt{x(x+y)^2}.$

ADDITION AND SUBTRACTION OF RADICALS.

Page 205.

2.
$$\begin{aligned}\sqrt{18} &= \sqrt{9} \times \sqrt{2} = 3\sqrt{2}\\ \sqrt{128} &= \sqrt{64} \times \sqrt{2} = 8\sqrt{2}\\ \sqrt{32} &= \sqrt{16} \times \sqrt{2} = \underline{4\sqrt{2}}\\ && 15\sqrt{2}\end{aligned}$$

3.
$$\begin{aligned}\sqrt{75} &= \sqrt{25} \times \sqrt{3} = 5\sqrt{3}\\ \sqrt{12} &= \sqrt{4} \times \sqrt{3} = 2\sqrt{3}\\ \sqrt{48} &= \sqrt{16} \times \sqrt{3} = \underline{4\sqrt{3}}\\ && 11\sqrt{3}\end{aligned}$$

4.
$$\begin{aligned}\sqrt[3]{32} &= \sqrt[3]{8} \times \sqrt[3]{4} = 2\sqrt[3]{4}\\ \sqrt[3]{108} &= \sqrt[3]{27} \times \sqrt[3]{4} = 3\sqrt[3]{4}\\ \sqrt[3]{256} &= \sqrt[3]{64} \times \sqrt[3]{4} = \underline{4\sqrt[3]{4}}\\ && 9\sqrt[3]{4}\end{aligned}$$

5.
$$\begin{aligned}3\sqrt{200} &= 3\sqrt{100} \times \sqrt{2} = 30\sqrt{2}\\ 4\sqrt{50} &= 4\sqrt{25} \times \sqrt{2} = 20\sqrt{2}\\ \sqrt{72} &= \sqrt{36} \times \sqrt{2} = \underline{6\sqrt{2}}\\ && 56\sqrt{2}\end{aligned}$$

6.
$$\begin{aligned}\sqrt[3]{162} &= \sqrt[3]{27} \times \sqrt[3]{6} = 3\sqrt[3]{6}\\ \sqrt[3]{384} &= \sqrt[3]{64} \times \sqrt[3]{6} = 4\sqrt[3]{6}\\ \sqrt[3]{750} &= \sqrt[3]{125} \times \sqrt[3]{6} = \underline{5\sqrt[3]{6}}\\ && 12\sqrt[3]{6}\end{aligned}$$

7.
$$\begin{array}{rcl}
3\sqrt{\tfrac{2}{3}} = 3\sqrt{\tfrac{6}{9}} = 3\sqrt{\tfrac{1}{9}} \times \sqrt{6} &=& \sqrt{6}\\
7\sqrt{\tfrac{27}{50}} = 7\sqrt{\tfrac{54}{100}} = 7\sqrt{\tfrac{9}{100}} \times \sqrt{6} &=& \underline{2\tfrac{1}{10}\sqrt{6}}\\
&& 3\tfrac{1}{10}\sqrt{6}
\end{array}$$

8.
$$\begin{array}{rcl}
\sqrt{24} = \sqrt{4} \times \sqrt{6} &=& 2\sqrt{6}\\
\sqrt{54} = \sqrt{9} \times \sqrt{6} &=& 3\sqrt{6}\\
\sqrt{294} = \sqrt{49} \times \sqrt{6} &=& \underline{7\sqrt{6}}\\
&& 12\sqrt{6}
\end{array}$$

9.
$$\begin{array}{rcl}
\sqrt{\tfrac{3}{4}} = \sqrt{\tfrac{1}{4}} \times \sqrt{3} &=& \tfrac{1}{2}\sqrt{3}\\
\sqrt{\tfrac{1}{3}} = \sqrt{\tfrac{3}{9}} = \sqrt{\tfrac{1}{9}} \times \sqrt{3} &=& \tfrac{1}{3}\sqrt{3}\\
\tfrac{1}{2}\sqrt{3} &=& \underline{\tfrac{1}{2}\sqrt{3}}\\
&& 1\tfrac{1}{3}\sqrt{3}
\end{array}$$

10.
$$\begin{array}{rcl}
\tfrac{1}{2}\sqrt{\tfrac{1}{2}} = \tfrac{1}{2}\sqrt{\tfrac{2}{4}} = \tfrac{1}{2}\sqrt{\tfrac{1}{4}} \times \sqrt{2} &=& \tfrac{1}{4}\sqrt{2}\\
\tfrac{3}{4}\sqrt{2} &=& \tfrac{3}{4}\sqrt{2}\\
\sqrt{\tfrac{1}{8}} = \sqrt{\tfrac{2}{16}} = \sqrt{\tfrac{1}{16}} \times \sqrt{2} &=& \underline{\tfrac{1}{4}\sqrt{2}}\\
&& 1\tfrac{1}{4}\sqrt{2}
\end{array}$$

11.
$$\begin{array}{rcl}
3\sqrt{242\,x^5y^5} = 3\sqrt{121\,x^4y^4} \times \sqrt{2\,xy} &=& 33\,x^2y^2\sqrt{2\,xy}\\
11\,xy\sqrt{2\,x^3y^3} = 11\,xy\sqrt{x^2y^2} \times \sqrt{2\,xy} &=& \underline{11\,x^2y^2\sqrt{2\,xy}}\\
&& 44\,x^2y^2\sqrt{2\,xy}
\end{array}$$

12.
$$\begin{array}{rcl}
\sqrt{108} = \sqrt{36} \times \sqrt{3} &=& 6\sqrt{3}\\
\sqrt{243} = \sqrt{81} \times \sqrt{3} &=& 9\sqrt{3}\\
\sqrt{300} = \sqrt{100} \times \sqrt{3} &=& \underline{10\sqrt{3}}\\
&& 25\sqrt{3}
\end{array}$$

13.
$$\begin{array}{rcl}
\sqrt{98\,x^2y^2} = \sqrt{49\,x^2y^2} \times \sqrt{2} &=& 7\,xy\sqrt{2}\\
\sqrt{128\,x^2y^2} = \sqrt{64\,x^2y^2} \times \sqrt{2} &=& 8\,xy\sqrt{2}\\
\sqrt{162\,x^2y^2} = \sqrt{81\,x^2y^2} \times \sqrt{2} &=& \underline{9\,xy\sqrt{2}}\\
&& 24\,xy\sqrt{2}
\end{array}$$

14.
$$\begin{array}{rcl}
\sqrt[3]{40} = \sqrt[3]{8} \times \sqrt[3]{5} &=& 2\sqrt[3]{5}\\
\sqrt[3]{135} = \sqrt[3]{27} \times \sqrt[3]{5} &=& 3\sqrt[3]{5}\\
\sqrt[3]{320} = \sqrt[3]{64} \times \sqrt[3]{5} &=& \underline{4\sqrt[3]{5}}\\
&& 9\sqrt[3]{5}
\end{array}$$

15. $\sqrt{24} + \sqrt{54} - \sqrt{96} = 2\sqrt{6} + 3\sqrt{6} - 4\sqrt{6} = \sqrt{6}.$

16. $\sqrt{12} + 3\sqrt{75} - 2\sqrt{27} = 2\sqrt{3} + 15\sqrt{3} - 6\sqrt{3} = 11\sqrt{3}.$

17. $5\sqrt{72} + 3\sqrt{18} - \sqrt{50} = 30\sqrt{2} + 9\sqrt{2} - 5\sqrt{2} = 34\sqrt{2}.$

18. $\sqrt{45\,a^3} + 5\sqrt{20\,a^3} - \sqrt{80\,a^3} = 3\,a\sqrt{5\,a} + 10\,a\sqrt{5\,a} - 4\,a\sqrt{5\,a} = 9\,a\sqrt{5\,a}.$

19. $\sqrt{8a^2y}+\sqrt{8b^2y}-\sqrt{50c^2y}=2a\sqrt{2y}+2b\sqrt{2y}-5c\sqrt{2y}$
$=(2a+2b-5c)\sqrt{2y}.$

20. $\sqrt[3]{128}+\sqrt[3]{686}-\sqrt[3]{54}=4\sqrt[3]{2}+7\sqrt[3]{2}-3\sqrt[3]{2}=8\sqrt[3]{2}.$

21. $\sqrt{192a}-\sqrt{75a}+\sqrt{12a}=8\sqrt{3a}-5\sqrt{3a}+2\sqrt{3a}=5\sqrt{3a}.$

22.
$$\begin{array}{rrr} 4\sqrt{3}= & & 4\sqrt{3} \\ 7\sqrt{\tfrac{4}{3}}= & 7\sqrt{\tfrac{4}{9}}\times 3= & \tfrac{14}{3}\sqrt{3} \\ -5\sqrt{\tfrac{4}{3}}= & -5\sqrt{\tfrac{4}{9}}\times 3= & -\tfrac{10}{3}\sqrt{3} \\ \hline & & 5\tfrac{1}{3}\sqrt{3} \end{array}$$

Page 206.

23. $\sqrt{\frac{a}{x^2}}+\sqrt{\frac{a}{y^2}}-\sqrt{\frac{a}{z^2}}=\frac{1}{x}\sqrt{a}+\frac{1}{y}\sqrt{a}-\frac{1}{z}\sqrt{a}=\left(\frac{1}{x}+\frac{1}{y}-\frac{1}{z}\right)\sqrt{a}.$

24. $\sqrt{\frac{a^5b^2}{c^2x^3}}-\sqrt{\frac{a^3c^2}{d^4x}}+\sqrt{\frac{ad^2y^4}{p^2x}}=\frac{a^2b}{cx}\sqrt{\frac{a}{x}}-\frac{ac}{d^2}\sqrt{\frac{a}{x}}+\frac{dy^2}{p}\sqrt{\frac{a}{x}}$
$=\left(\frac{a^2b}{cx^2}-\frac{ac}{d^2x}+\frac{dy^2}{px}\right)\sqrt{ax}.$

25. $\sqrt[4]{48}+\sqrt[4]{768}-\sqrt[4]{243}=2\sqrt[4]{3}+4\sqrt[4]{3}-3\sqrt[4]{3}=3\sqrt[4]{3}.$

26. $\sqrt{(x+y)^2z}+\sqrt{(x-y)^2z}=(x+y)\sqrt{z}+(x-y)\sqrt{z}=2x\sqrt{z}.$

27. $\sqrt{5}+\sqrt{20}-2\sqrt{\tfrac{1}{5}}=\sqrt{5}+2\sqrt{5}-\tfrac{2}{5}\sqrt{5}=2\tfrac{3}{5}\sqrt{5}.$

28. $\sqrt{\frac{x+y}{x-y}}+\sqrt{x^2-y^2}=\frac{1}{x-y}\sqrt{x^2-y^2}+\sqrt{x^2-y^2}=\frac{x-y+1}{x-y}\sqrt{x^2-y^2}.$

29. $\sqrt[5]{x^3y^6z^7}+\sqrt[5]{x^8yz^2}-\sqrt[5]{x^{13}y^{11}z^{12}}=yz\sqrt[5]{x^3yz^2}+x\sqrt[5]{x^3yz^2}-x^2y^2z^2\sqrt[5]{x^3yz^2}.$
$=(yz+x-x^2y^2z^2)\sqrt[5]{x^3yz^2}.$

30. $\sqrt[3]{256}+3\sqrt[3]{500}-2\sqrt[3]{108}=4\sqrt[3]{4}+15\sqrt[3]{4}-6\sqrt[3]{4}=13\sqrt[3]{4}.$

31. $\sqrt{288}-\sqrt{392}+\sqrt{450}=12\sqrt{2}-14\sqrt{2}+15\sqrt{2}=13\sqrt{2}.$

32. $\sqrt[3]{-24}+3\sqrt[3]{-375}+\sqrt[3]{-81}=2\sqrt[3]{-3}+15\sqrt[3]{-3}+3\sqrt[3]{-3}=20\sqrt[3]{-3}.$

33. $6\sqrt{\tfrac{3}{4}}+\tfrac{7}{4}\sqrt{\tfrac{1}{3}}-\sqrt{27}=3\sqrt{3}+\tfrac{7}{12}\sqrt{3}-3\sqrt{3}=\tfrac{7}{12}\sqrt{3}.$

34. $2\sqrt[3]{3x}+5\sqrt[6]{9x^2}-\sqrt[3]{27x^3}=2\sqrt[3]{3x}+5\sqrt[3]{3x}-\sqrt[3]{3x}=6\sqrt[3]{3x}.$

35. $\sqrt{2x^2}+4xy+2y^2=\sqrt{x^2}+2xy+y^2\times\sqrt{2}=(x+y)\sqrt{2}$;

$\sqrt{2x^2}-4xy+2y^2=\sqrt{x^2}-2xy+y^2\times\sqrt{2}$.

Hence, $(x+y)\sqrt{2}-(x-y)\sqrt{2}=2y\sqrt{2}$.

36. $\sqrt[4]{32x^4y^8}+\sqrt[4]{162x^4y^8}-\sqrt[4]{512x^4y^8}+\sqrt[4]{1250x^4y^8}=2xy^2\sqrt[4]{2}+3xy^2\sqrt[4]{2}$
$-4xy^2\sqrt[4]{2}+5xy^2\sqrt[4]{2}=6xy^2\sqrt[4]{2}$.

37. $3\sqrt{\frac{3a}{4b^2}}+2\sqrt{\frac{a}{3b^2}}-\sqrt{\frac{a}{27b^2}}=\frac{3}{2b}\sqrt{3a}+\frac{2}{3b}\sqrt{3a}-\frac{1}{9b}\sqrt{3a}=\frac{37}{18b}\sqrt{3a}$

38. $\sqrt{72}+\frac{3}{4}\sqrt{50}-\frac{1}{\sqrt{2}}=6\sqrt{2}+\frac{15}{4}\sqrt{2}-\frac{1}{2}\sqrt{2}=9\frac{1}{4}\sqrt{2}$.

MULTIPLICATION OF RADICALS.

Page 207.

14. $2\sqrt{3}\times3\sqrt[3]{2}=2\sqrt[6]{27}\times3\sqrt[6]{4}=6\sqrt[6]{108}$.

15. $3\sqrt{2}\times\sqrt[3]{3}=3\sqrt[6]{8}\times\sqrt[6]{9}=3\sqrt[6]{72}$.

16. $\sqrt{4}\times\sqrt[4]{5}=2\times\sqrt[4]{5}=2\sqrt[4]{5}$.

17. $3\sqrt{3}\times2\sqrt[3]{5}=3\sqrt[6]{27}\times2\sqrt[6]{25}=6\sqrt[6]{675}$.

18. $2\sqrt[3]{3}\times3\sqrt[4]{4}=2\sqrt[12]{81}\times3\sqrt[12]{64}=6\sqrt[12]{72^2}=6\sqrt[6]{72}$.

19. $3\sqrt{32}\times4\sqrt[3]{108}=3\sqrt[6]{32768}\times4\sqrt[6]{11664}=12\sqrt[6]{2\times24^6}=288\sqrt[6]{2}$.

23. $\sqrt[3]{x^2y}\times\sqrt[3]{2x^2y^2}\times\sqrt{3x^3y^2}=\sqrt[6]{x^4y^2}\times\sqrt[6]{4x^4y^4}\times\sqrt[6]{27x^9y^6}$
$=\sqrt[6]{108x^{17}y^{12}}=x^2y^2\sqrt[6]{108x^5}$.

24. $\sqrt[3]{3axy^2}\times\sqrt{axy}\times\sqrt{ax^2y}=\sqrt[6]{9a^2x^2y^4}\times\sqrt[6]{a^3x^3y^3}\times\sqrt[6]{a^3x^6y^3}$
$=\sqrt[6]{9a^8x^{11}y^{10}}=axy\sqrt[6]{9a^2x^5y^4}$.

25. $\sqrt{ab}\times2\sqrt{ac}\times\sqrt[3]{a^2b^2}=\sqrt[6]{a^3b^3}\times2\sqrt[6]{a^3c^3}\times\sqrt[6]{a^4b^4}=2\sqrt[6]{a^{10}b^7c^3}$
$=2ab\sqrt[6]{a^4bc^3}$.

26. $2\sqrt{2xy}\times3\sqrt[3]{x^2y}\times3\sqrt[3]{8xy^2}=2\sqrt[6]{8x^3y^3}\times3\sqrt[6]{x^4y^2}\times3\sqrt[6]{64x^2y^4}$
$=18\sqrt[6]{512x^9y^9}=18\sqrt[6]{64x^6y^6\times8x^3y^3}=36xy\sqrt{2xy}$.

Page 208.

30. $\sqrt[3]{\frac{2}{3}}\times\sqrt{\frac{1}{2}}\times\sqrt[3]{\frac{3}{4}}=\sqrt[6]{\frac{4}{9}}\times\sqrt[6]{\frac{1}{8}}\times\sqrt[6]{\frac{9}{16}}=\sqrt[6]{\frac{1}{32}}=\frac{1}{2}\sqrt[6]{2}$.

31. $\sqrt[3]{\frac{2}{5}}\times\sqrt[3]{\frac{2}{3}}\times\sqrt[2]{3}=\sqrt[6]{\frac{4}{25}}\times\sqrt[6]{\frac{8}{27}}\times\sqrt[6]{9}=\sqrt[6]{\frac{32}{75}}$.

32. $\sqrt[3]{\frac{5}{6}}\times\sqrt[3]{\frac{4}{3}}\times\sqrt{\frac{3}{2}}=\sqrt[6]{\frac{25}{36}}\times\sqrt[6]{\frac{16}{9}}\times\sqrt[6]{\frac{27}{8}}=\sqrt[6]{\frac{25}{6}}$.

33. $\sqrt[3]{\frac{7}{2}}\times\sqrt{\frac{5}{3}}\times\sqrt[3]{\frac{3}{10}}=\sqrt[6]{\frac{49}{4}}\times\sqrt[6]{\frac{125}{27}}\times\sqrt[6]{\frac{9}{100}}=\sqrt[6]{\frac{245}{48}}$.

35.
$$\begin{array}{l} \quad\ 3 + 4\sqrt{2} \\ \quad\ \underline{4 - 3\sqrt{2}} \\ 12 + 16\sqrt{2} \\ \quad\ \underline{-\ 9\sqrt{2} - 24} \\ 12 + 7\sqrt{2} - 24 = 7\sqrt{2} - 12. \end{array}$$

36.
$$\begin{array}{l} 4 + 3\sqrt{3} \\ \underline{5 + 2\sqrt{3}} \\ 20 + 15\sqrt{3} \\ \qquad\ \ \underline{8\sqrt{3} + 18} \\ 20 + 23\sqrt{3} + 18 = 38 + 23\sqrt{3}. \end{array}$$

37. This example and several that follow are the *product of the sum and difference* of two quantities, or the *square of their sum or difference;* hence the results may be written out and afterward simplified.

44.
$$\begin{array}{l} \quad a + \sqrt{ab} + b \\ \underline{\sqrt{a} - \sqrt{b}\qquad\qquad} \\ a\sqrt{a} + a\sqrt{b} + b\sqrt{a} \\ \underline{\qquad - a\sqrt{b} - b\sqrt{a} - b\sqrt{b}} \\ a\sqrt{a} \qquad\qquad\qquad - b\sqrt{b} \end{array}$$

46.
$$\begin{array}{l} x + x\sqrt{3} + 1 \\ \underline{x\sqrt{3} + 1\qquad\qquad\qquad\qquad\qquad} \\ x^2\sqrt{3} + 3x^2 + x\sqrt{3} + x + 1 \\ \underline{\qquad\qquad\qquad\quad x\sqrt{3}\qquad\quad} \\ x^2\sqrt{3} + 3x^2 + 2x\sqrt{3} + x + 1 \end{array}$$

DIVISION OF RADICALS.

Page 209.

11. $3\sqrt{2} \div \sqrt[3]{2} = 3\sqrt[6]{8} \div \sqrt[6]{4} = 3\sqrt[6]{2}.$

12. $\sqrt{3} \div \sqrt[3]{4} = \sqrt[6]{27} \div \sqrt[6]{16} = \sqrt[6]{\frac{27}{16}} = \frac{1}{2}\sqrt[6]{108}.$

13. $2\sqrt{5} \div \sqrt[3]{5} = 2\sqrt[6]{125} \div \sqrt[6]{25} = 2\sqrt[6]{5}.$

14. $3\sqrt{x^2z} \div 4\sqrt[3]{x^2z^2} = 3\sqrt[6]{x^6z^3} \div 4\sqrt[6]{x^4z^4} = \frac{3}{4}\sqrt[6]{x^2z^{-1}}.$

15. $\sqrt{4} \div \sqrt[3]{6} = \sqrt[6]{64} \div \sqrt[6]{36} = \sqrt[6]{\frac{8^2}{6^2}} = \sqrt[3]{\frac{8}{6}} = \sqrt[3]{\frac{4}{3}} = \frac{1}{3}\sqrt[3]{36}.$

16. $\sqrt{72} \div \sqrt[4]{64} = \sqrt{72} \div \sqrt{8} = \sqrt{9} = 3.$

17. $\sqrt[3]{12} \div \sqrt{8} = \sqrt[6]{144} \div \sqrt[6]{512} = \sqrt[6]{\frac{18}{64}} = \frac{1}{2}\sqrt[6]{18}.$

18. $3\sqrt[3]{18} \div \sqrt{6} = 3\sqrt[6]{324} \div \sqrt[6]{216} = 3\sqrt[6]{\frac{3}{2}} = \frac{3}{2}\sqrt[6]{96}.$

19. $6\sqrt[3]{ax} \div 2\sqrt{xy} = 6\sqrt[6]{a^2x^2} \div 2\sqrt[6]{x^3y^3} = 3\sqrt[6]{\frac{a^2}{xy^3}} = \frac{3}{xy}\sqrt[6]{a^2x^5y^3}.$

20. $\sqrt[3]{4x^2} \div \sqrt{2ax} = \sqrt[6]{16x^4} \div \sqrt[6]{8a^3x^3} = \sqrt[6]{\frac{2x}{a^3}} = \frac{1}{a}\sqrt[6]{2a^3x}.$

21. $\sqrt[3]{4xz} \div \sqrt[4]{2xz} = \sqrt[12]{256x^4z^4} \div \sqrt[12]{8x^3z^3} = \sqrt[12]{32xz}.$

22. $\sqrt[3]{3x^2y} \div \sqrt[5]{6x^3y^2} = \sqrt[15]{243x^{10}y^5} \div \sqrt[15]{216x^9y^6} = \sqrt[15]{\frac{243x}{216y}}.$

23. $2\sqrt[4]{x^2y^3z} \div 3\sqrt[3]{xy^2z^2} = 2\sqrt[12]{x^6y^9z^3} \div 3\sqrt[12]{x^4y^8z^8} = \frac{2}{3}\sqrt[12]{x^2yz^{-5}}.$

24. $\sqrt[6]{12a^3b^2c^5} \div \sqrt[4]{2a^2b^3c} = \sqrt[12]{144a^6b^4c^{10}} \div \sqrt[12]{8a^6b^9c^3} = \sqrt[12]{18b^{-5}c^7}.$

25. $3\sqrt[3]{a^4x^7y} \div \sqrt[4]{ax} = 3\sqrt[12]{a^{16}x^{28}y^4} \div \sqrt[12]{a^3x^3} = 3\sqrt[12]{a^{13}x^{25}y^4} = 3ax^2\sqrt[12]{axy^4}$

26.
$$\begin{array}{l|l} \sqrt{20}+\sqrt{12} & \underline{\sqrt{5}+\sqrt{3}} \\ \underline{\sqrt{20}+\sqrt{12}} & \underline{\sqrt{4}=2} \end{array}$$

27.
$$\begin{array}{l|l} 6+11\sqrt{2}+6 & \underline{3+\sqrt{2}} \\ \underline{6+\ \ 2\sqrt{2}} & \underline{2+3\sqrt{2}} \\ \qquad 9\sqrt{2}+6 & \\ \qquad \underline{9\sqrt{2}+6} & \end{array}$$

28.
$$\begin{array}{l|l} 8+10\sqrt{5}+15 & \underline{2+\sqrt{5}} \\ \underline{8+\ \ 4\sqrt{5}} & \underline{4+3\sqrt{5}} \\ \qquad 6\sqrt{5}+15 & \\ \qquad \underline{6\sqrt{5}+15} & \end{array}$$

Page 210.

29.
$$\begin{array}{l|l} 12+17\sqrt{6}+36 & \underline{3+2\sqrt{6}} \\ \underline{12+\ \ 8\sqrt{6}} & \underline{4+3\sqrt{6}} \\ \qquad 9\sqrt{6}+36 & \\ \qquad 9\sqrt{6}+36 & \end{array}$$

30.
$$\begin{aligned} & 12-2\sqrt{4}-\sqrt{48}\,\underline{|\sqrt{12}} \\ &= 12-4-4\sqrt{3}\,\underline{|2\sqrt{3}} \\ &= 8-4\sqrt{3}\,\underline{|2\sqrt{3}} \\ &= \frac{8}{2\sqrt{3}}-2 \\ &= \frac{4}{\sqrt{3}}-2 \\ &= \frac{4\sqrt{3}}{3}-2;\ \text{or}\ \frac{4}{3}\sqrt{3}-2 \end{aligned}$$

31.
$$\begin{array}{l|l} a\sqrt[3]{a}-\sqrt[3]{a^2b}-b\sqrt[3]{a^2b}+b\sqrt[3]{b^2} & \underline{\sqrt[3]{a^2}-\sqrt[3]{b}} \\ \underline{a\sqrt[3]{a}-\sqrt[3]{a^2b}\qquad\qquad\qquad\qquad} & \underline{\sqrt[3]{a^2}-b\sqrt[3]{b}} \\ \qquad\qquad -b\sqrt[3]{a^2b}+b\sqrt[3]{b^2} & \\ \qquad\qquad \underline{-b\sqrt[3]{a^2b}+b\sqrt[3]{b^2}} & \end{array}$$

32.

$$\begin{array}{l|l}
x + y - z + 2\sqrt{xy} & \underline{\sqrt{x} + \sqrt{y} - \sqrt{z}} \\
\underline{x - \sqrt{xz} \quad + \sqrt{xy}} & \underline{\sqrt{x} + \sqrt{z} + \sqrt{y}} \\
\sqrt{xz} + y - z + \sqrt{xy} & \\
\underline{\sqrt{xz} + \sqrt{yz} - z} & \\
\sqrt{xy} + y - \sqrt{zy} & \\
\underline{\sqrt{xy} + y - \sqrt{yz}} &
\end{array}$$

RATIONALIZATION.

Page 214.

17. $\dfrac{7}{\sqrt{2}} = \dfrac{7\sqrt{2}}{2} = \dfrac{7 \times 1.414+}{2} = 4.949+.$

18. $\dfrac{16}{\sqrt{3}} = \dfrac{16\sqrt{3}}{3} = \dfrac{16 \times 1.73205}{3} = 9.237+.$

19. $\dfrac{4}{\sqrt{2}} = \dfrac{4\sqrt{2}}{2} = \dfrac{4 \times 1.414+}{2} = 2.828+.$

20. $\dfrac{4}{\sqrt{300}} = \dfrac{4\sqrt{300}}{300} = \dfrac{4 \times 17.3205}{300} = .2309+.$

21. $\dfrac{5}{\sqrt{147}} = \dfrac{5\sqrt{147}}{147} = \dfrac{5 \times 12.1243}{147} = .4123+.$

22. $\dfrac{1}{3\sqrt{2}} = \dfrac{\sqrt{2}}{6} = \dfrac{1.4142}{6} = .2357+.$

23. $\dfrac{\sqrt{3} - \sqrt{2}}{\sqrt{3} - \sqrt{2}} = \dfrac{3 - 2}{3 - 2} = 1.$

24. $\dfrac{\sqrt{8} + \sqrt{5}}{\sqrt{8} - \sqrt{5}} = \dfrac{8 + 2\sqrt{40} + 5}{8 - 5} = \dfrac{13 + 4\sqrt{10}}{3} = 8.549+.$

25. $\dfrac{4}{\sqrt{3} + \sqrt{2}} = \dfrac{4(\sqrt{3} - \sqrt{2})}{3 - 2} = \dfrac{4(.31784+)}{1} = 1.2713+.$

BINOMIAL SURDS.

Page 216.

4. Let $x + y =$ square root of $5 + 2\sqrt{6}$;

then, $x^2 + y^2 = 5;$ (1)

and, $2xy = 2\sqrt{6}.$ (2)

(1) + (2), $(x + y)^2 = 5 + 2\sqrt{6}.$ (3)

(1) − (2), $(x - y)^2 = 5 - 2\sqrt{6}.$ (4)

(3) × (4), $(x^2 - y^2)^2 = 25 - 24 = 1.$ (5)

Square root (5), $x^2 - y^2 = 1.$ (6)

(1) + (6), $2x^2 = 6;$

$x^2 = 3;$

$x = \sqrt{3};$

and, $y = \sqrt{2}.$

$$\therefore \sqrt{5 + 2\sqrt{6}} = \sqrt{3} + \sqrt{2}.$$

5. Let $x + y =$ square root of $9 + 2\sqrt{8};$

then, $x^2 + y^2 = 9;$ (1)

and, $2xy = 2\sqrt{8}.$ (2)

(1) + (2), $(x + y)^2 = 9 + 2\sqrt{8}.$ (3)

(1) − (2), $(x - y)^2 = 9 - 2\sqrt{8}.$ (4)

(3) × (4), $(x^2 - y^2)^2 = 81 - 32 = 49.$ (5)

Square root (5), $x^2 - y^2 = 7.$ (6)

(1) + (6), $2x^2 = 16;$

$x^2 = 8;$

$x = 2\sqrt{2};$

and, $y = 1.$

$$\therefore \sqrt{9 + 2\sqrt{8}} = 2\sqrt{2} + 1.$$

6. Let $x - y =$ square root of $8 - 2\sqrt{15};$

then, $x^2 + y^2 = 8;$ (1)

and, $-2xy = -2\sqrt{15}.$ (2)

(1) + (2), $(x - y)^2 = 8 - 2\sqrt{15}.$ (3)

(1) − (2), $(x + y)^2 = 8 + 2\sqrt{15}.$ (4)

(3) × (4), $(x^2 - y^2)^2 = 64 - 60 = 4.$ (5)

Square root (5), $x^2 - y^2 = 2.$ (6)

(1) + (6), $2x^2 = 10;$

$x^2 = 5;$

$x = \sqrt{5};$

and, $y = \sqrt{3}.$

$$\therefore \sqrt{8 - 2\sqrt{15}} = \sqrt{5} - \sqrt{3}.$$

7. Let $x - y =$ square root of $11 - 6\sqrt{2};$

then, $x^2 + y^2 = 11;$ (1)

and, $-2xy = -6\sqrt{2}.$ (2)

(1) + (2), $(x - y)^2 = 11 - 6\sqrt{2}.$ (3)

(1) − (2), $(x + y)^2 = 11 + 6\sqrt{2}$. (4)

(3) × (4), $(x^2 - y^2)^2 = 121 - 72 = 49$. (5)

Square root (5), $x^2 - y^2 = 7$. (6)

(1) + (6), $2x^2 = 18$;

$x^2 = 9$;

$x = 3$;

and, $y = \sqrt{2}$.

$$\therefore \sqrt{16 - 6\sqrt{2}} = 3 - \sqrt{2}.$$

8. Let $x + y =$ square root of $6a + 2a\sqrt{5}$;

then, $x^2 + y^2 = 6a$; (1)

and, $2xy = 2a\sqrt{5}$. (2)

(1) + (2), $(x + y)^2 = 6a + 2a\sqrt{5}$. (3)

(1) − (2), $(x - y)^2 = 6a - 2a\sqrt{5}$. (4)

(3) × (4), $(x^2 - y^2) = 36a^2 - 20a^2 = 16a^2$. (5)

Square root (5), $x^2 - y^2 = 4a$. (6)

(1) + (6), $2x^2 = 10a$;

$x^2 = 5a$;

$x = \sqrt{5a}$;

and, $y = \sqrt{a}$.

$$\therefore \sqrt{6a + 2a\sqrt{5}} = \sqrt{5a} + \sqrt{a}.$$

9. Let $x - y =$ square root of $5 - \sqrt{24}$;

then, $x^2 + y^2 = 5$; (1)

and, $-2xy = -\sqrt{24}$. (2)

(1) + (2), $(x - y)^2 = 5 - \sqrt{24}$. (3)

(1) − (2), $(x + y)^2 = 5 + \sqrt{24}$. (4)

(3) × (4), $(x^2 - y^2)^2 = 25 - 24 = 1$. (5)

Square root (5), $x^2 - y^2 = 1$. (6)

(1) + (6), $2x^2 = 6$;

$x^2 = 3$;

$x = \sqrt{3}$;

and, $y = \sqrt{2}$.

$$\therefore \sqrt{5 - \sqrt{24}} = \sqrt{3} - \sqrt{2}.$$

10. Let $x + y =$ square root of $15 + 4\sqrt{14}$;

then, $x^2 + y^2 = 15$; (1)

and, $2xy = 4\sqrt{14}$. (2)

(1) + (2), $(x + y)^2 = 15 + 4\sqrt{14}$. (3)

(1) − (2), $(x - y)^2 = 15 - 4\sqrt{14}$. (4)

(3) × (4), $(x^2 - y^2)^2 = 225 - 224 = 1$. (5)

Square root (5), $x^2 - y^2 = 1$. (6)

(1) + (6), $2x^2 = 16$;

$x^2 = 8$;

$x = 2\sqrt{2}$;

and, $y = \sqrt{7}$.

$\therefore \sqrt{15 + 4\sqrt{14}} = 2\sqrt{2} + \sqrt{7}$.

11. Let $x + y =$ square root of $28 + 5\sqrt{12}$;

then, $x^2 + y^2 = 28$; (1)

and, $2xy = 5\sqrt{12}$. (2)

(1) + (2), $(x + y)^2 = 28 + 5\sqrt{12}$. (3)

(1) − (2), $(x - y)^2 = 28 - 5\sqrt{12}$. (4)

(3) × (4), $(x^2 - y^2)^2 = 784 - 300 = 484$. (5)

Square root (5), $x^2 - y^2 = 22$. (6)

(1) + (6), $2x^2 = 50$;

$x^2 = 25$;

$x = 5$;

and, $y = \sqrt{3}$.

$\therefore \sqrt{28 + 5\sqrt{12}} = 5 + \sqrt{3}$.

12. Let $x + y =$ square root of $13 + 2\sqrt{30}$;

then, $x^2 + y^2 = 13$; (1)

and, $2xy = 2\sqrt{30}$. (2)

(1) + (2), $(x + y)^2 = 13 + 2\sqrt{30}$. (3)

(1) − (2), $(x - y)^2 = 13 - 2\sqrt{30}$. (4)

(3) × (4), $(x^2 - y^2)^2 = 169 - 120 = 49$. (5)

Square root (5), $x^2 - y^2 = 7$. (6)

(1) + (6), $2x^2 = 20$;

$x^2 = 10$;

$x = \sqrt{10}$;

and, $y = \sqrt{3}$.

$\therefore \sqrt{13 + 2\sqrt{30}} = \sqrt{10} + \sqrt{3}$.

13. Let $x - y =$ square root of $7 - 2\sqrt{10}$;

then, $x^2 + y^2 = 7$; (1)

and, $-2xy = -2\sqrt{10}$. (2)

(1)+(2), $(x - y)^2 = 7 - 2\sqrt{10}$. (3)

(1)−(2), $(x + y)^2 = 7 + 2\sqrt{10}$. (4)

(3)×(4), $(x^2 - y^2)^2 = 49 - 40 = 9$. (5)

Square root (5), $x^2 - y^2 = 3$. (6)

(1)+(6), $2x^2 = 10$;

$x^2 = 5$;

$x = \sqrt{5}$;

and, $y = \sqrt{2}$.

$\therefore \sqrt{7 - 2\sqrt{10}} = \sqrt{5} - \sqrt{2}$.

14. Let $x + y =$ square root of $11 + 2\sqrt{30}$;

then, $x^2 + y^2 = 11$; (1)

and, $2xy = 2\sqrt{30}$. (2)

(1)+(2), $(x + y)^2 = 11 + 2\sqrt{30}$. (3)

(1)−(2), $(x - y)^2 = 11 - 2\sqrt{30}$. (4)

(3)×(4), $(x^2 - y^2)^2 = 121 - 120 = 1$. (5)

Square root (5), $x^2 - y^2 = 1$. (6)

(1)+(6), $2x^2 = 12$;

$x^2 = 6$;

$x = \sqrt{6}$;

and, $y = \sqrt{5}$.

$\therefore \sqrt{11 + 2\sqrt{30}} = \sqrt{6} + \sqrt{5}$.

15. Let $x - y =$ square root of $2m - 2\sqrt{m^2 - n^2}$;

then, $x^2 + y^2 = 2m$; (1)

and, $-2xy = -2\sqrt{m^2 - n^2}$. (2)

(1)+(2), $(x - y)^2 = 2m - 2\sqrt{m^2 - n^2}$. (3)

(1)−(2), $(x + y)^2 = 2m + 2\sqrt{m^2 - n^2}$. (4)

(3)×(4). $(x^2 - y^2)^2 = 4m^2 - 4m^2 + 4n^2 = 4n^2$. (5)

Square root (5), $x^2 - y^2 = 2n$. (6)

(1)+(6), $2x^2 = 2m + 2n$;

$x^2 = m + n$;

$x = \sqrt{m + n}$;

and, $y = \sqrt{m - n}$.

$\therefore \sqrt{2m - 2\sqrt{m^2 - n^2}} = \sqrt{m + n} - \sqrt{m - n}$.

16. Let $x - y =$ square root of $15 - 2\sqrt{56}$;

then, $x^2 + y^2 = 15$; (1)

and, $-2xy = -2\sqrt{56}$. (2)

(1) + (2), $(x - y)^2 = 15 - 2\sqrt{56}$. (3)

(1) − (2), $(x + y)^2 = 15 + 2\sqrt{56}$. (4)

(3) × (4), $(x^2 - y^2)^2 = 225 - 224 = 1$. (5)

Square root (5), $x^2 - y^2 = 1$. (6)

(1) + (6), $2x^2 = 16$;

$x^2 = 8$;

$x = 2\sqrt{2}$;

and, $y = \sqrt{7}$.

$\therefore \sqrt{15 - 2\sqrt{56}} = 2\sqrt{2} - \sqrt{7}$.

17. Let $x + y =$ square root of $14 + 3\sqrt{20}$;

then, $x^2 + y^2 = 14$; (1)

and, $2xy = 3\sqrt{20}$. (2)

(1) + (2), $(x + y)^2 = 14 + 3\sqrt{20}$. (3)

(1) − (2), $(x - y)^2 = 14 - 3\sqrt{20}$. (4)

(3) × (4), $(x^2 - y^2)^2 = 196 - 180 = 16$. (5)

Square root (5), $x^2 - y^2 = 4$. (6)

(1) + (6), $2x^2 = 18$;

$x^2 = 9$;

$x = 3$;

and, $y = \sqrt{5}$.

$\therefore \sqrt{14 + 3\sqrt{20}} = 3 + \sqrt{5}$.

18. Let $x + y =$ square root of $6 + \sqrt{35}$;

then, $x^2 + y^2 = 6$; (1)

and, $2xy = \sqrt{35}$. (2)

(1) + (2), $(x + y)^2 = 6 + \sqrt{35}$. (3)

(1) − (2), $(x - y)^2 = 6 - \sqrt{35}$. (4)

(3) × (4), $(x^2 - y^2)^2 = 36 - 35 = 1$. (5)

Square root (5), $x^2 - y^2 = 1$. (6)

(1) + (6), $2x^2 = 7$;

$x^2 = \frac{7}{2}$;

$x = \frac{1}{2}\sqrt{14}$;

and, $y = \frac{1}{2}\sqrt{10}$.

$\therefore \sqrt{6 + \sqrt{35}} = \frac{1}{2}(\sqrt{14} + \sqrt{10})$.

19. Let $x+y=$ square root of $28+10\sqrt{3}$;

then, $x^2+y^2=28;$ (1)

and, $2xy=10\sqrt{3}.$ (2)

(1) + (2), $(x+y)^2=28+10\sqrt{3}.$ (3)

(1) − (2), $(x-y)^2=28-10\sqrt{3}.$ (4)

(3) × (4), $(x^2-y^2)^2=784-300=484.$ (5)

Square root (5), $x^2-y^2=22.$ (6)

(1) + (6), $2x^2=50;$

$x^2=25;$

$x=5;$

and, $y=\sqrt{3}.$

$$\therefore \sqrt{28+10\sqrt{3}}=5+\sqrt{3}.$$

20. Let $x+y=$ square root of $m^2+n+2m\sqrt{n}$;

then, $x^2+y^2=m^2+n;$ (1)

and, $2xy=2m\sqrt{n}.$ (2)

(1) + (2), $(x+y)^2=m^2+n+2m\sqrt{n}.$ (3)

(1) − (2), $(x-y)^2=m^2+n-2m\sqrt{n}.$ (4)

(3) × (4), $(x^2-y^2)^2=(m^2+n)^2-4m^2n=(m^2-n)^2.$ (5)

Square root (5), $x^2-y^2=m^2-n.$ (6)

(1) + (6), $2x^2=2m^2;$

$x^2=m^2;$

$x=m;$

and, $y=\sqrt{n}.$

$$\therefore \sqrt{m^2+n+2m\sqrt{n}}=m+\sqrt{n}.$$

21. Let $x+y=$ square root of $25+2\sqrt{156}$;

then, $x^2+y^2=25;$ (1)

and, $2xy=2\sqrt{156}.$ (2)

(1) + (2), $(x+y)^2=25+2\sqrt{156}.$ (3)

(1) − (2), $(x-y)^2=25-2\sqrt{156}.$ (4)

(3) × (4), $(x^2-y^2)^2=625-624=1.$ (5)

Square root (5), $x^2-y^2=1.$ (6)

(1) + (6), $2x^2=26;$

$x^2=13;$

$x=\sqrt{13};$

and, $y=2\sqrt{3}.$

$$\therefore \sqrt{25+2\sqrt{156}}=\sqrt{13}+2\sqrt{3}.$$

REVIEW.

2. $\left(\frac{2x}{5}+\frac{3y}{7}\right)^3=\left(\frac{2x}{5}\right)^3+3\left(\frac{2x}{5}\right)^2\left(\frac{3y}{7}\right)+3\left(\frac{2x}{5}\right)\left(\frac{3y}{7}\right)^2+\left(\frac{3y}{7}\right)^3$
$=\frac{8x^3}{125}+\frac{36x^2y}{175}+\frac{54xy^2}{245}+\frac{27y^3}{343}.$

3. $(2a-3b)^4=(2a)^4-4(2a)^3(3b)+6(2a)^2(3b)^2-4(2a)(3b)^3+(3b)^4$
$=16a^4-96a^3b+216a^2b^2-216ab^3+81b^4.$

4. $\left(\frac{x}{4}+\frac{y}{3}\right)^5=\left(\frac{x}{4}\right)^5+5\left(\frac{x}{4}\right)^4\left(\frac{y}{3}\right)+10\left(\frac{x}{4}\right)^3\left(\frac{y}{3}\right)^2+10\left(\frac{x}{4}\right)^2\left(\frac{y}{3}\right)^3+5\left(\frac{x}{4}\right)\left(\frac{y}{3}\right)^4$
$+\left(\frac{y}{3}\right)^5=\frac{x^5}{1024}+\frac{5x^4y}{768}+\frac{5x^3y^2}{288}+\frac{5x^2y^3}{216}+\frac{5xy^4}{324}+\frac{y^5}{243}.$

5. $(3a+4b)^5=(3a)^5+5(3a)^4(4b)+10(3a)^3(4b)^2+10(3a)^2(4b)^3$
$+5(3a)(4b)^4+(4b)^5=243a^5+1620a^4b+4320a^3b^2$
$+5760a^2b^3+3840ab^4+1024b^5.$

6. $\left(\frac{a}{2}-\frac{b}{3}\right)^3=\left(\frac{a}{2}\right)^3-3\left(\frac{a}{2}\right)^2\left(\frac{b}{3}\right)+3\left(\frac{a}{2}\right)\left(\frac{b}{3}\right)^2-\left(\frac{b}{3}\right)^3=\frac{a^3}{8}-\frac{a^2b}{4}+\frac{ab^2}{6}-\frac{b^3}{27}.$

10. $(1+2x+3y+z)^2=1+4x^2+9y^2+z^2+4x+6y+2z+12xy+4xz+6yz.$

Page 217.

11. $\sqrt{128}-2\sqrt{50}+\sqrt{72}-\sqrt{78}=8\sqrt{2}-10\sqrt{2}+6\sqrt{2}-\sqrt{78}=4\sqrt{2}-\sqrt{78}.$

12. $6\sqrt[6]{4a^2}+2\sqrt[3]{2a}+\sqrt[9]{2a^3}=6\sqrt[3]{2a}+2\sqrt[3]{2a}+\sqrt[9]{2a^3}=8\sqrt[3]{2a}+\sqrt[9]{2a^3}.$

13. $2\sqrt{3}-\frac{1}{2}\sqrt{12}+4\sqrt{27}-2\sqrt{\frac{3}{16}}=2\sqrt{3}-\sqrt{3}+12\sqrt{3}-\frac{1}{2}\sqrt{3}=12\frac{1}{2}\sqrt{3}.$

14. $\sqrt[4]{16}+\sqrt[3]{81}-\sqrt[3]{-512}+\sqrt[3]{192}-7\sqrt[6]{9}=2+3\sqrt[3]{3}+8+4\sqrt[3]{3}-7\sqrt[3]{3}=10.$

15. $\sqrt{48ab^2}+b\sqrt{75a}+\sqrt{3a(a-9b)^2}=4b\sqrt{3a}+5b\sqrt{3a}+(a-9b)\sqrt{3a}$
$=a\sqrt{3a}.$

16.

$$\begin{array}{r|l}
 & 9x-24x^{\frac{1}{2}}y^{\frac{2}{3}}+16y^{\frac{4}{3}}+12x^{\frac{1}{2}}-16y^{\frac{2}{3}}+4\ \underline{\big|\,3x^{\frac{1}{2}}-4y^{\frac{2}{3}}+2} \\
 & \underline{9x} \\
6x^{\frac{1}{2}}-4y^{\frac{2}{3}} & -24x^{\frac{1}{2}}y^{\frac{2}{3}}+16y^{\frac{4}{3}} \\
 & \underline{-24x^{\frac{1}{2}}y^{\frac{2}{3}}+16y^{\frac{4}{3}}} \\
6x^{\frac{1}{2}}-8y^{\frac{2}{3}}+2 & 12x^{\frac{1}{2}}-16y^{\frac{2}{3}}+4 \\
 & \underline{12x^{\frac{1}{2}}-16y^{\frac{2}{3}}+4}
\end{array}$$

17.

$$\begin{array}{rl}
 & 4a^{\frac{4}{3}}+4a^{\frac{5}{3}}-15a^2-8a^{\frac{7}{3}}+16a^{\frac{8}{3}}\,\big|\,\underline{2a^{\frac{2}{3}}+a-4a^{\frac{4}{3}}}\\
 & \underline{4a^{\frac{4}{3}}}\\
4a^{\frac{2}{3}}+a & \big|\ 4a^{\frac{5}{3}}-15a^2\\
 & \big|\ \underline{4a^{\frac{5}{3}}+\quad a^2}\\
4a^{\frac{2}{3}}+2a-4a^{\frac{4}{3}} & \big|\ -16a^2-8a^{\frac{7}{3}}+16a^{\frac{8}{3}}\\
 & \big|\ \underline{-16a^2-8a^{\frac{7}{3}}+16a^{\frac{8}{3}}}
\end{array}$$

18.

$$\begin{array}{rl}
 & x^{\frac{4}{3}}-4xy^{\frac{1}{3}}+6x^{\frac{2}{3}}y^{\frac{2}{3}}-4x^{\frac{1}{3}}y+y^{\frac{4}{3}}\,\big|\,\underline{x^{\frac{2}{3}}-2x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}}\\
 & \underline{x^{\frac{4}{3}}}\\
2x^{\frac{2}{3}}-2x^{\frac{1}{3}}y^{\frac{1}{3}} & \big|\ -4xy^{\frac{1}{3}}+6x^{\frac{2}{3}}y^{\frac{2}{3}}\\
 & \big|\ \underline{-4xy^{\frac{1}{3}}+4x^{\frac{2}{3}}y^{\frac{2}{3}}}\\
2x^{\frac{2}{3}}-4x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}} & \big|\ 2x^{\frac{2}{3}}y^{\frac{2}{3}}-4x^{\frac{1}{3}}y+y^{\frac{4}{3}}\\
 & \big|\ \underline{2x^{\frac{2}{3}}y^{\frac{2}{3}}-4x^{\frac{1}{3}}y+y^{\frac{4}{3}}}
\end{array}$$

19.

$$\begin{array}{rl}
 & 9x-12x^{\frac{5}{6}}+10x^{\frac{2}{3}}-28x^{\frac{1}{2}}+17x^{\frac{1}{3}}-8x^{\frac{1}{6}}+16\,\big|\,\underline{3x^{\frac{1}{2}}-2x^{\frac{1}{3}}+x^{\frac{1}{6}}-4}\\
 & \underline{9x}\\
6x^{\frac{1}{2}}-2x^{\frac{1}{3}} & \big|\ -12x^{\frac{5}{6}}+10x^{\frac{2}{3}}\\
 & \big|\ \underline{-12x^{\frac{5}{6}}+\ 4x^{\frac{2}{3}}}\\
6x^{\frac{1}{2}}-4x^{\frac{1}{3}}+x^{\frac{1}{6}} & \big|\ 6x^{\frac{2}{3}}-28x^{\frac{1}{2}}+17x^{\frac{1}{3}}\\
 & \big|\ \underline{6x^{\frac{2}{3}}-\ 4x^{\frac{1}{2}}+\quad x^{\frac{1}{3}}}\\
6x^{\frac{1}{2}}-4x^{\frac{1}{3}}+2x^{\frac{1}{6}}-4 & \big|\ -24x^{\frac{1}{2}}+16x^{\frac{1}{3}}-8x^{\frac{1}{6}}+16\\
 & \big|\ \underline{-24x^{\frac{1}{2}}+16x^{\frac{1}{3}}-8x^{\frac{1}{6}}+16}
\end{array}$$

20.

$$\begin{array}{rl}
 & x^6-3x^5+5x^3-3x-1\,\big|\,\underline{x^2-x-1}\\
 & \underline{x^6}\\
\text{Trial divisor, }3x^4 & \big|\ -3x^5+5x^3\\
3x^4-3x^3+x^2 & \big|\ \underline{-3x^5+3x^4-x^3}\\
\text{Trial divisor, }3x^4-6x^3+3x^2 & \big|\ -3x^4+6x^3-3x-1\\
\underline{3x^4-6x^3+3x+1} & \big|\ \underline{-3x^4+6x^3-3x-1}
\end{array}$$

21.

$$\begin{array}{rl}
 & x^{\frac{6}{5}}-6x+15x^{\frac{4}{5}}-20x^{\frac{3}{5}}+15x^{\frac{2}{5}}-6x^{\frac{1}{5}}+1\,\big|\,\underline{x^{\frac{2}{5}}-2x^{\frac{1}{5}}+1}\\
 & \underline{x^{\frac{6}{5}}}\\
\text{Trial divisor, }3x^{\frac{4}{5}} & \big|\ -6x+15x^{\frac{4}{5}}-20x^{\frac{3}{5}}\\
3x^{\frac{4}{5}}-6x^{\frac{3}{5}}+4x^{\frac{2}{5}} & \big|\ \underline{-6x+12x^{\frac{4}{5}}-\ 8x^{\frac{3}{5}}}\\
\text{Trial divisor, }3x^{\frac{4}{5}}-12x^{\frac{3}{5}}+4x^{\frac{2}{5}} & \big|\ 3x^{\frac{4}{5}}-12x^{\frac{3}{5}}+15x^{\frac{2}{5}}-6x^{\frac{1}{5}}+1\\
\underline{3x^{\frac{4}{5}}-12x^{\frac{3}{5}}+15x\text{[illegible]}6x^{\frac{1}{5}}+1} & \big|\ \underline{3x^{\frac{4}{5}}-12x^{\frac{3}{5}}+15x^{\frac{2}{5}}-6x^{\frac{1}{5}}+1}
\end{array}$$

22.

$$x^{\frac{1}{2}}+3x^{\frac{5}{12}}y^{\frac{1}{12}}-3x^{\frac{1}{3}}y^{\frac{1}{6}}-11x^{\frac{1}{4}}y^{\frac{1}{4}}+6x^{\frac{1}{6}}y^{\frac{1}{3}}+12x^{\frac{1}{12}}y^{\frac{5}{12}}-8y^{\frac{1}{2}}\ \Big|\ \underline{x^{\frac{1}{6}}+x^{\frac{1}{12}}y^{\frac{1}{12}}-2y^{\frac{1}{6}}}$$

$$\underline{x^{\frac{1}{2}}}$$

$$\text{Trial divisor, } 3x^{\frac{1}{3}}\ \Big|\ 3x^{\frac{5}{12}}y^{\frac{1}{12}}-3x^{\frac{1}{3}}y^{\frac{1}{6}}-11x^{\frac{1}{4}}y^{\frac{1}{4}}$$

$$3x^{\frac{1}{3}}+3x^{\frac{3}{12}}y^{\frac{1}{12}}+x^{\frac{2}{12}}y^{\frac{2}{12}}\ \Big|\ \underline{3x^{\frac{5}{12}}y^{\frac{1}{12}}+3x^{\frac{1}{3}}y^{\frac{1}{6}}+x^{\frac{1}{4}}y^{\frac{1}{4}}}$$

$$\text{Trial divisor, } 3x^{\frac{1}{3}}+6x^{\frac{1}{4}}y^{\frac{1}{12}}+3x^{\frac{1}{6}}y^{\frac{1}{6}}\ \Big|\ -6x^{\frac{1}{3}}y^{\frac{1}{6}}-12x^{\frac{1}{4}}y^{\frac{1}{4}}+6x^{\frac{1}{6}}y^{\frac{1}{3}}+12x^{\frac{1}{12}}y^{\frac{5}{12}}-8y^{\frac{1}{2}}$$

$$\underline{-3x^{\frac{1}{3}}+6x^{\frac{1}{4}}y^{\frac{1}{12}}-3x^{\frac{1}{6}}y^{\frac{1}{6}}-6x^{\frac{1}{12}}y^{\frac{1}{4}}+4y^{\frac{2}{6}}\ \Big|\ -6x^{\frac{1}{3}}y^{\frac{1}{6}}-12x^{\frac{1}{4}}y^{\frac{1}{4}}+6x^{\frac{1}{6}}y^{\frac{1}{3}}+12x^{\frac{1}{12}}y^{\frac{5}{12}}-8y^{\frac{1}{2}}}$$

23.

$$a^{\frac{1}{3}}x^{-1}+a^{-\frac{1}{3}}x\ \Big|\ \underline{a^{\frac{1}{9}}x^{-\frac{1}{3}}+\tfrac{1}{3}a^{-\frac{5}{9}}x^{\frac{5}{3}}-\tfrac{1}{9}a^{-\frac{11}{9}}x^{\frac{11}{3}}, \text{ etc.}}$$

$$\underline{a^{\frac{1}{3}}x^{-1}}$$

$$\text{Trial divisor, } 3a^{\frac{2}{9}}x^{-\frac{2}{3}}\ \Big|\ a^{-\frac{1}{3}}x$$

$$3a^{\frac{2}{9}}x^{-\frac{2}{3}}+a^{-\frac{4}{9}}x^{\frac{4}{3}}+\tfrac{1}{9}a^{-\frac{10}{9}}x^{\frac{10}{3}}\ \Big|\ \underline{a^{-\frac{1}{3}}x+\tfrac{1}{3}a^{-1}x^{3}+\tfrac{1}{27}a^{-\frac{5}{3}}x^{5}}$$

$$\text{Trial divisor, } 3a^{\frac{2}{9}}x^{-\frac{2}{3}}+\tfrac{2}{3}a^{-\frac{4}{9}}x^{\frac{4}{3}}+\tfrac{1}{3}a^{-\frac{10}{9}}x^{\frac{10}{3}}\ \Big|\ -\tfrac{1}{3}a^{-1}x^{3}-\tfrac{1}{27}a^{-\frac{5}{3}}x^{5}$$

$$\underline{3a^{\frac{2}{9}}x^{-\frac{2}{3}}+\tfrac{2}{3}a^{-\frac{4}{9}}x^{\frac{4}{3}}-\tfrac{2}{3}a^{-\frac{10}{9}}x^{\frac{10}{3}}-\tfrac{1}{9}a^{-\frac{16}{9}}x^{\frac{16}{3}}+\tfrac{1}{81}a^{-\frac{22}{9}}x^{\frac{22}{3}}\ \Big|\ -\tfrac{1}{3}a^{-1}x^{3}-\tfrac{2}{27}a^{-\frac{15}{9}}x^{\frac{15}{3}}+\tfrac{2}{81}a^{-\frac{21}{9}}x^{\frac{21}{3}}+\tfrac{1}{81}a^{-\frac{27}{9}}x^{\frac{27}{3}}-\tfrac{1}{729}a^{-\frac{33}{9}}x^{\frac{33}{3}}}$$

24.

$$\sqrt{a}+\sqrt{a-x}+\sqrt{x}$$

$$\underline{\sqrt{a}-\sqrt{a-x}+\sqrt{x}}$$

$$a+\sqrt{a}\sqrt{a-x}+\sqrt{ax}+\ \ x-\sqrt{x}\sqrt{a-x}$$

$$\underline{-a-\sqrt{a}\sqrt{a-x}+\sqrt{ax}+\ \ x+\sqrt{x}\sqrt{a-x}}$$

$$2\sqrt{ax}+2x$$

25. These [illegible]ctors are the sum and difference of two quantities, and the multiplicatio[illegible]ay be written out.

26. Write t[illegible]terms with fractional exponents.

$$
\begin{array}{l}
2x^{\frac{5}{3}} - x^{\frac{1}{3}} - 3x^{-1} \\
\underline{2x - 3x^{-\frac{1}{3}} - x^{-\frac{5}{3}}} \\
4x^{\frac{8}{3}} - 2x^{\frac{4}{3}} - 6 \\
\qquad - 6x^{\frac{4}{3}} + 3 + 9x^{-\frac{4}{3}} \\
\underline{\qquad\qquad\quad - 2 + \quad x^{-\frac{4}{3}} + 3x^{-\frac{8}{3}}} \\
4x^{\frac{8}{3}} - 8x^{\frac{4}{3}} - 5 + 10x^{-\frac{4}{3}} + 3x^{-\frac{8}{3}}
\end{array}
$$

Page 218.

27.

$$
\begin{array}{r|l}
m^2 + mn + n^2 & \underline{m + \sqrt{mn} + n} \\
\underline{m^2 + m\sqrt{n} + mn} & \underline{m - \sqrt{mn} + n} \\
- m\sqrt{mn} + n^2 & \\
\underline{- m\sqrt{mn} - mn - n\sqrt{mn}} & \\
mn + n\sqrt{mn} + n^2 & \\
\underline{mn + n\sqrt{mn} + n^2} &
\end{array}
$$

28.

$$
\begin{array}{r|l}
x^4 + y^4 & \underline{x^2 + xy\sqrt{2} + y^2} \\
\underline{x^4 + x^3y\sqrt{2} + x^2y^2} & \underline{x^2 - xy\sqrt{2} + y^2} \\
- x^3y\sqrt{2} - x^2y^2 + y^4 & \\
\underline{- x^3y\sqrt{2} - 2x^2y^2 - xy^3\sqrt{2}} & \\
x^2y^2 + xy^3\sqrt{2} + y^4 & \\
\underline{x^2y^2 + xy^3\sqrt{2} + y^4} &
\end{array}
$$

29.
$$\frac{a - 7a^{\frac{1}{2}}}{a - 5\sqrt{a} - 14} \div \left(1 + \frac{2}{\sqrt{a}}\right)^{-1} = \frac{a - 7a^{\frac{1}{2}}}{a - 5\sqrt{a} - 14} \div \frac{1}{\frac{\sqrt{a} + 2}{\sqrt{a}}}$$

$$= \frac{a - 7a^{\frac{1}{2}}}{a - 5\sqrt{a} - 14} \times \frac{\sqrt{a} + 2}{\sqrt{a}} = \frac{a^{\frac{3}{2}} - 5a - 14a^{\frac{1}{2}}}{a^{\frac{3}{2}} - 5a - 14a^{\frac{1}{2}}} = 1.$$

30. $(\sqrt[3]{2y^2} + \sqrt[6]{x^3y})^2 = \sqrt[3]{4y^4} + 2\sqrt[6]{4x^3y^5} + \sqrt[6]{x^6y^2} = y\sqrt[3]{4y} + 2\sqrt[6]{4x^3y^5} + x\sqrt[3]{y}.$

31. $(2 + a^2\sqrt{2x + y})^2 = 4 + 4a^2\sqrt{2x + y} + a^4(2x + y).$

32. $(3\sqrt{8} + 5\sqrt{3})^2 = 9 \times 8 + 30\sqrt{24} + 25 \times 3 = 72 + 60\sqrt{6} + 75 = 147 + 60\sqrt{6}.$

33. $(\sqrt{1 - x^2} + x)^2 = 1 - x^2 + 2x\sqrt{1 - x^2} + x^2 = 1 + 2x\sqrt{1 - x^2}.$

34. $(\sqrt[6]{12})^3 = \sqrt[6]{1728} = \sqrt{12} = 3.464+.$

35. $(\sqrt[3]{a^2x})^5 = \sqrt[3]{a^{10}x^5} = a^3x\sqrt[3]{ax^2}.$

36. $(a\sqrt[3]{a})^2 = a^2\sqrt[3]{a^2}.$

37. $(3\sqrt[6]{3})^3 = 27\sqrt[6]{3^3} = 27\sqrt{3}.$

38. $\sqrt[5]{\sqrt{64}} = \sqrt[5]{8} =$

		8	1.515+
		1	
Trial divisor,	50000	700000	
	50000	659375	
	25000		
	6250		
	625		
Complete divisor,	131875		
Trial divisor,	2531250000	4062500000	
	33750000	2565225751	
	225000		
	750		
	1		
Complete divisor,	2565225751		
Trial divisor,	25173480050000	149727424900000	
	172147550000	126730992846875	
	570025000		
	943750		
	625		
Complete divisor,	25346198569375		

39. $\sqrt[3]{\sqrt[5]{54x^3}} = \sqrt[5]{\sqrt[3]{2 \times 27x^3}} = \sqrt[5]{3x\sqrt[3]{2}}.$

40. In this and the seven following examples, multiply both terms of the fractions by the denominators with the second sign changed.

$$\frac{2+\sqrt{3}}{3-\sqrt{3}} = \frac{6+5\sqrt{3}+3}{9-3} = \frac{9+5\sqrt{3}}{6}.$$

41. $\dfrac{3+\sqrt{5}}{\sqrt{3}-\sqrt{2}} = \dfrac{3\sqrt{3}+3\sqrt{2}+\sqrt{15}+\sqrt{10}}{(3-2)=1}.$

43. $\dfrac{4+3\sqrt{2}}{3-2\sqrt{2}} = \dfrac{12+17\sqrt{2}+12}{9-8} = \dfrac{24+17\sqrt{2}}{1}.$

44. $\dfrac{2\sqrt{3}+3\sqrt{2}}{5+2\sqrt{6}} = \dfrac{3\sqrt{2}-2\sqrt{3}}{1}.$

45. $\dfrac{10\sqrt{6}-2\sqrt{7}}{3\sqrt{6}+2\sqrt{7}} = \dfrac{180-26\sqrt{42}+28}{54-28} = \dfrac{208-26\sqrt{42}}{26} = \dfrac{8-\sqrt{42}}{1}.$

46. $\dfrac{a+\sqrt{a^2-x^2}}{a-\sqrt{a^2-x^2}}=\dfrac{a^2+2a\sqrt{a^2-x^2}+a^2-x^2}{a^2-a^2+x^2}=\dfrac{2a^2+2a\sqrt{a^2-x^2}-x^2}{x^2}$

47. $\dfrac{\sqrt{m^2+1}-\sqrt{m^2-1}}{\sqrt{m^2+1}+\sqrt{m^2-1}}=\dfrac{m^2+1-2\sqrt{m^4-1}+m^2-1}{m^2+1-m^2+1}$

$=\dfrac{2m^2-2\sqrt{m^4-1}}{2}=\dfrac{m^2-\sqrt{m^4-1}}{1}.$

48. $\dfrac{3}{\sqrt{5}}=\dfrac{3\sqrt{5}}{5}=\dfrac{3\times 2.236}{5}=1.341.$

49. $\dfrac{5\sqrt{2}}{\sqrt{3}}=\dfrac{5\sqrt{6}}{3}=\dfrac{5\times 2.449}{3}=4.081.$

50. $\dfrac{\sqrt{7}+\sqrt{3}}{\sqrt{7}-\sqrt{3}}=\dfrac{7+2\sqrt{21}+3}{7-3}=\dfrac{10+2\sqrt{21}}{4}=\dfrac{5+\sqrt{21}}{2}=4.791.$

51. $\dfrac{2\sqrt{5}-\sqrt{3}}{3\sqrt{5}+2\sqrt{3}}=\dfrac{30-7\sqrt{15}+6}{45-12}=\dfrac{36-7\sqrt{15}}{33}=\dfrac{36-27.104}{33}=.269.$

52. Let $x-y=$ square root of $9-4\sqrt{5}$;

then, $x^2+y^2=9;$ (1)

and, $-2xy=-4\sqrt{5}.$ (2)

(1) + (2), $(x-y)^2=9-4\sqrt{5}.$ (3)

(1) − (2), $(x+y)^2=9+4\sqrt{5}.$ (4)

(3) × (4), $(x^2-y^2)^2=81-80=1.$ (5)

Square root (5), $x^2-y^2=1.$ (6)

(1) + (6), $2x^2=10;$

$x^2=5;$

$x=\sqrt{5};$

and, $y=2.$

$\therefore\ \sqrt{9-4\sqrt{5}}=\sqrt{5}-2.$

53. Let $x-y=$ square root of $43-15\sqrt{8}$;

then, $x^2+y^2=43;$ (1)

and, $-2xy=-15\sqrt{8}.$ (2)

(1) + (2), $(x-y)^2=43-15\sqrt{8}.$ (3)

(1) − (2), $(x+y)^2=43+15\sqrt{8}.$ (4)

(3) × (4), $(x^2-y^2)^2=1849-1800=49.$ (5)

Square root (5), $x^2 - y^2 = 7.$ (6)

(1) + (6), $2x^2 = 50;$

$x^2 = 25;$

$x = 5;$

and, $y = 3\sqrt{2}.$

$$\therefore \sqrt{43 - 15\sqrt{8}} = 5 - 3\sqrt{2}.$$

54. Let $x - y =$ square root of $24 - 2\sqrt{63};$

then, $x^2 + y^2 = 24;$ (1)

and, $-2xy = -2\sqrt{63}.$ (2)

(1) + (2), $(x - y)^2 = 24 - 2\sqrt{63}.$ (3)

(1) − (2), $(x + y)^2 = 24 + 2\sqrt{63}.$ (4)

(3) × (4), $(x^2 - y^2)^2 = 576 - 252 = 324.$ (5)

Square root (5), $x^2 - y^2 = 18.$ (6)

(1) + (6), $2x^2 = 42;$

$x^2 = 21;$

$x = \sqrt{21};$

and, $y = \sqrt{3}.$

$$\therefore \sqrt{24 - 2\sqrt{63}} = \sqrt{21} - \sqrt{3}.$$

55. Let $x - y =$ square root of $22 - 4\sqrt{30};$

then, $x^2 + y^2 = 22;$ (1)

and, $-2xy = -4\sqrt{30}.$ (2)

(1) + (2), $(x - y)^2 = 22 - 4\sqrt{30}.$ (3)

(1) − (2). $(x + y)^2 = 22 + 4\sqrt{30}.$ (4)

(3) × (4), $(x^2 - y^2)^2 = 484 - 480 = 4.$ (5)

Square root (5), $x^2 - y^2 = 2.$ (6)

(1) + (6), $2x^2 = 24;$

$x^2 = 12;$

$x = 2\sqrt{3};$

and, $y = \sqrt{10}.$

$$\therefore \sqrt{22 - 4\sqrt{30}} = 2\sqrt{3} - \sqrt{10}.$$

56. Let $x - y = \text{square root of } 47 - 6\sqrt{10};$

then, $x^2 + y^2 = 47;$ (1)

and, $-2xy = -6\sqrt{10}.$ (2)

(1) + (2), $(x-y)^2 = 47 - 6\sqrt{10}.$ (3)

(1) − (2), $(x+y)^2 = 47 + 6\sqrt{10}.$ (4)

(3) × (4), $(x^2 - y^2)^2 = 2209 - 360 = 1849.$ (5)

Square root (5), $x^2 - y^2 = 43.$ (6)

(1) + (6), $2x^2 = 90;$

$x^2 = 45;$

$x = 3\sqrt{5};$

and, $y = \sqrt{2}.$

$$\therefore \sqrt{47 - 6\sqrt{10}} = 3\sqrt{5} - \sqrt{2}.$$

57. Let $x - y = \text{square root of } 23 - \sqrt{528};$

then, $x^2 + y^2 = 23;$ (1)

and, $-2xy = -\sqrt{528}.$ (2)

(1) + (2), $(x-y)^2 = 23 - \sqrt{528}.$ (3)

(1) − (2), $(x+y)^2 = 23 + \sqrt{528}.$ (4)

(3) × (4), $(x^2 - y^2)^2 = 529 - 528 = 1.$ (5)

Square root (5), $x^2 - y^2 = 1.$ (6)

(1) + (6), $2x^2 = 24;$

$x^2 = 12;$

$x = 2\sqrt{3};$

and, $y = \sqrt{11}.$

$$\therefore \sqrt{23 - \sqrt{528}} = 2\sqrt{3} - \sqrt{11}.$$

RADICAL EQUATIONS.

Page 221.

8. Cubing, $4x - 16 = 8;$

whence, $x = 6.$

9. Squaring, $x + 9 = 36;$

whence, $x = 27.$

10. Transposing (Sug. 1), $\sqrt[3]{2x+3}=3;$
whence, $x=12.$

11. Transposing (Sug. 1), $\sqrt{x^2-9}=9-x.$
Squaring, $x^2-9=81-18x+x^2;$
whence, $x=5.$

12. Transposing (Sug. 1), $\sqrt{x^2-11}=x-1.$
Squaring, $x^2-11=x^2-2x+1;$
whence, $x=6.$

13. Transposing (Sug. 1), $\sqrt{16+x}=8-\sqrt{x}.$
Squaring, $16+x=64-16\sqrt{x}+x.$
Transposing, etc., $16\sqrt{x}=48.$
Dividing by coefficient and squaring, $x=9.$

14. Squaring, $x-16=64-16\sqrt{x}+x.$
Transposing, etc., $16\sqrt{x}=80.$
Dividing by coefficient and squaring, $x=25.$

15. Squaring, $x-21=x-2\sqrt{x}+1.$
Transposing, etc., $2\sqrt{x}=22.$
Dividing by coefficient and squaring, $x=121.$

16. Transposing and squaring, $x+5=25-10\sqrt{x}+x.$
Transposing, etc., $10\sqrt{x}=20.$
Dividing by coefficient and squaring, $x=4.$

17. Transposing and squaring, $x-15=225-30\sqrt{x}+x.$
Transposing, etc., $30\sqrt{x}=240.$
Dividing by coefficient and squaring, $x=64.$

18. Transposing, $2\sqrt{x}-\sqrt{2}=\sqrt{4x-22}.$
Squaring, $4x-4\sqrt{2x}+2=4x-22.$
Transposing, etc., $-4\sqrt{2x}=-24.$
Dividing by coefficient, squaring, etc., $x=18.$

19. Clearing of fractions, $\sqrt{x^2-9x}+x-9=36.$
Transposing, etc., $\sqrt{x^2-9x}=45-x.$
Squaring, $x^2-9x=2025-90x+x^2;$
whence, $x=25.$

20. Clearing, $\sqrt{x^2-4x}+x-4=4.$

Transposing, etc., $\sqrt{x^2-4x}=8-x.$

Squaring, $x^2-4x=64-16x+x^2;$

whence, $x=5\frac{1}{3}.$

21. Clearing, $\sqrt{8x+x^2}+8+x=12.$

Transposing and squaring, $8x+x^2=16-8x+x^2.$

Transposing, etc., $x=1.$

22. Dividing the numerator of the first fraction by its denominator:

$$\sqrt{3x}-1-\frac{\sqrt{3x}-1}{2}=4.$$

Clearing, $2\sqrt{3x}-2-\sqrt{3x}+1=8.$

Transposing, etc., $\sqrt{3x}=9;$

whence, $x=27.$

23. Transposing and squaring,

$$x^2-\sqrt{1-x}=1-2x+x^2.$$

Transposing and squaring, $1-x=1-4x+4x^2.$

Transposing, etc., $x=\frac{3}{4}.$

24. Squaring, $1+x\sqrt{x^2+12}=1+2x+x^2.$

Transposing and dividing by x, $\sqrt{x^2+12}=2+x.$

Squaring, $x^2+12=4+4x+x^2.$

Transposing, etc., $x=2.$

25. Clearing, $a-x=a-x\sqrt{a-x}.$

Transposing and dividing by x, $\sqrt{a-x}=1;$

whence, $a-x=1$, and $x=a-1.$

26. Dividing the numerator of the first fraction by its denominator.

$$\sqrt{ax}-1=\frac{\sqrt{ax}-1}{2}+4.$$

Clearing, transposing, etc., $\sqrt{ax}-1=8;$

whence, $\sqrt{ax}=9$, and $x=\frac{81}{a}.$

27. Reducing the first member to a mixed quantity,

$$1+\frac{24}{\sqrt{x}+4}=\frac{32}{\sqrt{x}+6}+1.$$

Transposing and dividing by 8, $\frac{3}{\sqrt{x}+4}=\frac{4}{\sqrt{x}+6}.$

Clearing, $3\sqrt{x}+18=4\sqrt{x}+16.$

Transposing, etc., $x=4.$

28. Changing each fraction to a mixed quantity,

$$1+\frac{12}{\sqrt{x}+4}=1+\frac{20}{\sqrt{x}+12}.$$

Transposing, etc., $\frac{12}{\sqrt{x}+4}=\frac{20}{\sqrt{x}+12}.$

Dividing by 4, $\frac{3}{\sqrt{x}+4}=\frac{5}{\sqrt{x}+12}.$

Clearing, $3\sqrt{x}+36=5\sqrt{x}+20.$

Transposing, etc., $2\sqrt{x}=16;$

whence, $x=64.$

29. Clearing, $\sqrt{-x^2+2x-1}=\sqrt{-x^2+8x-15}.$

Squaring, $-x^2+2x-1=-x^2+8x-15.$

Transposing, etc., $6x=14;$

whence, $x=2\frac{1}{3}.$

30. Changing each fraction to a mixed quantity,

$$1-\frac{10}{\sqrt{x}+7}=1-\frac{5}{\sqrt{x}+1}.$$

Canceling and dividing by -5, $\frac{2}{\sqrt{x}+7}=\frac{1}{\sqrt{x}+1}.$

Clearing, $2\sqrt{x}+2=\sqrt{x}+7;$

whence, $x=25.$

31. Reducing each fraction to a mixed quantity,

$$1-\frac{2}{\sqrt{x}-6}=1-\frac{6}{\sqrt{x}+2}.$$

Transposing and dividing by 2, $\frac{3}{\sqrt{x}+2}=\frac{1}{\sqrt{x}-6}.$

Clearing, $3\sqrt{x}-18=\sqrt{x}+2.$

Transposing, etc., $x=100.$

Page 222.

32. Reducing each fraction to a mixed quantity:

$$1+\frac{20}{3\sqrt{2x}-10}=1+\frac{20}{\sqrt{2x}-4}.$$

Transposing and dividing by 20, $\dfrac{1}{3\sqrt{2x}-10}=\dfrac{1}{\sqrt{2x}-4}.$

Clearing, $\sqrt{2x}-4=3\sqrt{2x}-10;$

whence, $2\sqrt{2x}=6$, and $x=4\frac{1}{2}$.

33. Transposing, $\sqrt{x+8}=\sqrt{x-8}+2\sqrt{2}.$

Squaring, $x+8=x-8+4\sqrt{2x-16}+8.$

Transposing and dividing by 4, $\sqrt{2x-16}=2.$

Squaring, etc., $x=10.$

34. Clearing, $ab-b\sqrt{a^2-x^2}=x.$

Transposing and squaring, $a^2b^2-b^2x^2=x^2-2abx+a^2b^2.$

Transposing, etc., $b^2x^2+x^2=2abx.$

Dividing by x, $b^2x+x=2ab;$

whence, $x=\dfrac{2ab}{b^2+1}.$

35. Squaring, $\dfrac{1}{x}-2+x=1+x.$

Transposing, etc., $\dfrac{1}{x}=3;$

whence, $x=\frac{1}{3}.$

36. Rationalizing the denominator:

$$\frac{a+x+2\sqrt{ax+x^2}+x}{a}=b.$$

Clearing and transposing, $2\sqrt{ax+x^2}=ab-a-2x.$

Squaring, $\cancel{4ax}+4x^2=a^2b^2+a^2+x^2-2a^2b-4abx+\cancel{4ax}.$

Transposing, etc., $4abx=a^2b^2-2a^2b+a^2.$

Dividing by a, $4bx=ab^2-2ab+a$, or $a(b-1)^2;$

whence, $x=\dfrac{a(b-1)^2}{4b}.$

37. Rationalizing the denominator:

$$\frac{4x+1+4\sqrt{4x^2+x}+4x}{1}=9.$$

Transposing, etc., and dividing by 4, $\sqrt{4x^2+x}=2-2x.$

Squaring, $4x^2+x=4-8x+4x^2.$

Transposing, etc., $x=\frac{4}{9}.$

38. Rationalizing the denominator:

$$\frac{5+x+2\sqrt{25-x^2}+5-x}{2x}=2.$$

Clearing and transposing, $2\sqrt{25-x^2}=4x-10.$

Dividing by 2 and squaring, $25-x^2=4x^2-20x+25;$

whence, $x=4.$

39. Rationalizing the denominator:

$$\frac{a+2\sqrt{a^2-ax}+a-x}{x}=a.$$

Clearing and transposing, $2\sqrt{a^2-ax}=x-2a+ax.$

Squaring, $4a^2-4ax=x^2+\cancel{4a^2}+a^2x^2-\cancel{4ax}+2ax^2-4a^2x.$

Transposing, etc., $x^2+2ax^2+a^2x^2=4a^2x.$

Dividing by x, and factoring, $x(1+a)^2=4a^2;$

$$x=\frac{4a^2}{(1+a)^2}.$$

40. Clearing, $a+\sqrt{x}=ab-b\sqrt{x}.$

Transposing, $(b+1)\sqrt{x}=a(b-1);$

whence, $\sqrt{x}=\frac{a(b-1)}{b+1};$

and, $x=\frac{a^2(b-1)^2}{(b+1)^2}.$

41. Rationalizing the denominator:

$$\frac{b^2-2b\sqrt{b^2-x}+b^2-x}{x}=a.$$

Clearing and transposing, $2b\sqrt{b^2-x}=2b^2-x-ax.$

Squaring, $4b^4-4b^2x=4b^4+x^2+a^2x^2-4b^2x-4ab^2x+2ax^2.$

Transposing, etc., $x^2+2ax^2+a^2x^2=4ab^2x.$

Dividing by x, $x+2ax+a^2x=4ab^2.$

Factoring, $x(1+2a+a^2)=4ab^2;$

whence, $x=\frac{4ab^2}{(1+a)^2}.$

42. Clearing, $6x - 3\sqrt{2x - 4x^2} = 2x + \sqrt{2x - 4x^2}.$

Transposing, $4x = 4\sqrt{2x - 4x^2}.$

Dividing by 4, and squaring, $x^2 = 2x - 4x^2.$

Transposing, and dividing by x, $x = \frac{2}{5}.$

43. Reduce the fraction to an entire quantity:

$$\sqrt{2x + 1} = 2 - \sqrt{2x}.$$

Squaring, $2x + 1 = 4 - 4\sqrt{2x} + 2x.$

Transposing, etc., $4\sqrt{2x} = 3.$

Squaring, $32x = 9;$

whence, $x = \frac{9}{32}.$

44. Clearing, $\sqrt{4x + 3} + 2\sqrt{x - 1} = 5\sqrt{4x + 3} - 10\sqrt{x - 1}.$

Transposing, etc., $12\sqrt{x - 1} = 4\sqrt{4x + 3}.$

Dividing by 4, and squaring, $9x - 9 = 4x + 3;$

whence, $x = 2\frac{2}{5}.$

45. In first fraction, divide numerator by denominator:

$$\sqrt{x} + \sqrt{3} = \frac{\sqrt{x} + \sqrt{3}}{2} + 2\sqrt{3}.$$

Clearing, $2\sqrt{x} + 2\sqrt{3} = \sqrt{x} + \sqrt{3} + 4\sqrt{3}.$

Transposing, etc., $\sqrt{x} = 3\sqrt{3}.$

Squaring, $x = 27.$

46. Clearing, $2x + 5\sqrt{x} - 12 = 2x + 11\sqrt{x} - 30.$

Transposing, etc., $6\sqrt{x} = 18.$

Dividing by 6, and squaring, $x = 9.$

47. Transposing and squaring,

$$x^2 - 12x - 20 = 64 - 16\sqrt{x^2 + 4x + 12} + x^2 + 4x + 12.$$

Transposing, etc., $16\sqrt{x^2 + 4x + 12} = 16x + 96.$

Dividing by 16, and squaring, $x^2 + 4x + 12 = x^2 + 12x + 36.$

Transposing, etc., $8x = -24;$

whence, $x = -3.$

48. In first fraction, divide numerator by denominator:

$$\sqrt{x} - \sqrt{a} = \frac{\sqrt{x} - \sqrt{a}}{3} + 2\sqrt{a}.$$

Clearing, $3\sqrt{x} - 3\sqrt{a} = \sqrt{x} - \sqrt{a} + 6\sqrt{a}.$

Transposing, etc., $2\sqrt{x} = 8\sqrt{a}.$

Dividing by 2, and squaring, $x = 16a.$

49. Squaring, $\cancel{x}+a+2\sqrt{x^2+(a+b)x+ab}+\cancel{x}+b=4x+a+3b.$

Transposing, etc., $\sqrt{x^2+(a+b)x+ab}=x+b.$

Squaring, $x^2+(a+b)x+ab=x^2+2bx+b^2.$

Transposing, etc., $(a-b)x=b^2-ab;$

whence, $x=-b.$

50. Clearing, $2\sqrt{x+1}-2\sqrt{x-1}=\sqrt{x+1}+\sqrt{x-1}.$

Transposing, etc., $\sqrt{x+1}=3\sqrt{x-1}.$

Squaring, $x+1=9x-9;$

whence, $x=\frac{5}{4}.$

51. Clearing, $c\sqrt{a^2+x^2}+cx=b\sqrt{a^2+x^2}-bx.$

Transposing, etc., $(b-c)\sqrt{a^2+x^2}=(b+c)x.$

Squaring, $(b-c)^2(a^2+x^2)=(b+c)^2x^2.$

Transposing, $-4bcx^2=-(b-c)^2a^2.$

Dividing by coefficient and extracting square root,

$$x=\frac{(b-c)a}{2\sqrt{bc}}.$$

52. Rationalizing the denominator:

$$\frac{(a+x+\sqrt{2ax+x^2})^2}{a^2}=b.$$

Extracting square root, $\frac{a+x+\sqrt{2ax+x^2}}{a}=\sqrt{b}.$

Clearing and transposing, $\sqrt{2ax+x^2}=-a-x+a\sqrt{b}.$

Squaring, $2ax+x^2=a^2+x^2+a^2b+2ax-2a^2\sqrt{b}-2ax\sqrt{b}.$

Transposing, etc., $2ax\sqrt{b}=a^2(1-2\sqrt{b}+b);$

whence, $x=\frac{a(1-\sqrt{b})^2}{2\sqrt{b}}.$

QUADRATIC EQUATIONS.

Page 225.

15. Clearing, $3x^2+24x+48+3x^2-24x+48=10x^2-160.$

Transposing, etc., $4x^2=256;$

whence, $x=\pm 8.$

16. Clearing, $3+3x+3-3x=8-8x^2.$

Transposing, etc., $8x^2=2;$

whence, $x=\pm\frac{1}{2}.$

17. Clearing, $x^2 + 9x = 9x + 18.$

Transposing, etc., $x = \pm 3\sqrt{2}.$

18. Clearing, $4x^2 - 16 = x^2 - 4.$

Transposing, etc., $x = \pm 2.$

19. Clearing and squaring, $x^2 - a^2 = a^2;$

whence, $x = \pm a\sqrt{2}.$

20. Clearing and transposing, $x\sqrt{x^2 + 3} = 3 - x^2.$

Squaring and transposing, $9x^2 = 9;$

whence, $x = \pm 1.$

21. Clearing, $15 = x\sqrt{x^2 + 5} + x^2 + 5.$

Transposing and squaring, $x^4 + 5x^2 = x^4 - 20x^2 + 100;$

whence, $x = \pm 2.$

22. Transposing and squaring, $x^2 + 2\sqrt{1 - x} = 1 - 2x + x^2.$

Canceling x^2 and squaring, $4 - 4x = 1 - 4x + 4x^2;$

whence, $x = \pm\sqrt{\frac{3}{4}},$ or $\pm\frac{1}{2}\sqrt{3}.$

23. Clearing, $x^2 - 2mx + m^2 + x^2 + 2mx + m^2 = 4x^2 - 4m^2.$

Transposing, etc., $2x^2 = 6m^2;$

whence, $x = \pm m\sqrt{3}.$

24. Clearing,

$$x^2 + 2ax + a^2 + x^2 - 2ax + a^2 - ax^2 - 2a^2x - a^3 - ax^2 + 2a^2x - a^3 = 2ax^2 - 2a^3$$

Canceling, transposing, etc., $2ax^2 - x^2 = a^2;$

whence, $x^2 = \frac{a^2}{2a - 1};$

$$x = \pm\frac{a}{\sqrt{2a - 1}}.$$

25. Clearing, $\sqrt{25 + 16x} + \sqrt{25 - 16x} = 8.$

Squaring and uniting, $\sqrt{625 - 256x^2} = 7.$

Squaring and uniting again, $256x^2 = 576;$

whence, $x^2 = \frac{144}{64};$

$$x = \pm 1\tfrac{1}{2}.$$

26. Clearing, $x\sqrt{x^2 - 14} - x^2 + 14 = 5.$

Transposing and squaring, $x^4 - 14x^2 = x^4 - 18x^2 + 81.$

Canceling, uniting, etc., $x^2 = \frac{81}{4};$

whence, $x = \pm 4\frac{1}{2}.$

27. Expanding terms, canceling, etc., $x^2 = 36$;
whence, $x = \pm 6$.

28. Clearing, $6x - 3 - 4x + 7 = 4x^2 + 2x$.
Canceling, uniting, etc., $4x^2 = 4$;
whence, $x = \pm 1$.

29. Expanding, $x^3 + 4x + x^2 + 4 = x^3 - 3x^2 + 3x - 1 + x + 13$.
Canceling, transposing, etc., $4x^2 = 8$;
whence, $x = \pm\sqrt{2}$.

30. Clearing, $8x^4 + 8x^2 - 6 - 42x^2 + 28 = 8x^4 - 16x^2 + 6$.
Canceling, transposing, etc., $18x^2 = 16$;
whence, $x^2 = \frac{8}{9}$;
$x = \pm\frac{2}{3}\sqrt{2}$.

31. Squaring, $a + x = x + \sqrt{x^2 - b^2}$.
Canceling x and squaring, $a^2 = x^2 - b^2$;
whence, $x = \pm\sqrt{a^2 + b^2}$.

32. Rationalizing denominator,

$$\frac{x^2 + 1 - 2\sqrt{x^4 - 1} + x^2 - 1}{2} = \frac{1}{2}.$$

Clearing, transposing, etc., $2x^2 - 1 = 2\sqrt{x^4 - 1}$.
Squaring, $4x^4 - 4x^2 + 1 = 4x^4 - 4$;
whence, $x = \pm\sqrt{\frac{5}{4}}$, or $\pm\frac{1}{2}\sqrt{5}$.

33. Clearing, $2x^2 - 2x\sqrt{2 - x^2} + 2x^2 + 2x\sqrt{2 - x^2} = ax^2 - 2a + ax^2$.
Transposing, etc., $2ax^2 - 4x^2 = 2a$;
whence, $x^2 = \dfrac{a}{a - 2}$;
and, $x = \pm\sqrt{\dfrac{a}{a - 2}}$.

34. Clearing,

$$1 - x^2 + (1 + x)\sqrt{1 + x^2} = 2a + 2a\sqrt{1 + x^2} - 1 + x^2 - (1 - x)\sqrt{1 + x^2}.$$

Transposing, etc., $1 - x^2 + \sqrt{1 + x^2} = a + a\sqrt{1 + x^2}$;
or, $(1 - a)\sqrt{1 + x^2} = x^2 + a - 1$.
Squaring, $1 - 2a + a^2 + x^2 - 2ax^2 + a^2x^2 = a^2 + x^4 + 1 + 2ax^2 - 2a - 2x^2$.
Transposing, etc., $3x^2 - 4ax^2 + a^2x^2 = x^4$;
whence, $x^2 = 3 - 4a + a^2$;
and, $x = \pm\sqrt{3 - 4a + a^2}$, or $\pm\sqrt{(2 - a)^2 - 1}$.

16 03 5

3 9460

5760

15) 1560

QUADRATIC EQUATIONS.—PROBLEMS.

Page 226.

5. Let $3x =$ one number;

then, $5x =$ the other number;

and, $9x^2 + 25x^2 = 3400;$

whence, $x = \pm 10.$

Hence, $3x = \pm 30$ } the numbers.

and, $5x = \pm 50$ }

9460

5760

15) 1560

6. Let $x =$ father's age;

then, $\frac{x}{4} =$ son's age;

and, $x^2 - \frac{x^2}{16} = 960;$

whence, $x = 32$, the father's age;

and, $\frac{x}{4} = 8$, the son's age.

7. Let $x =$ principal;

then, $\frac{6x}{100} =$ interest for 1 year;

and, $\frac{2}{3}$ of $\frac{6x}{100}$, or $\frac{x}{25} =$ interest for 8 months.

Hence, $\frac{x^2}{25} = 900;$

whence, $x = 150$ dollars, the principal.

8. Let $2x =$ side of one room in feet;

then, $3x =$ side of other room in feet.

Hence, $\frac{9x^2}{9} - \frac{4x^2}{9} = 20;$

whence, $x = 6;$

and, $2x = 12$ feet } sides of rooms.

$3x = 18$ feet }

11. Let $6\frac{1}{2} + x =$ one number;

then, $6\frac{1}{2} - x =$ the other number;

and, $\frac{169}{4} - x^2 = 42;$

whence, $x = \frac{1}{2};$

and, $6\frac{1}{2} + x = 7$, one number;

$6\frac{1}{2} - x = 6$, the other number.

Page 227.

12. Let $10 + x =$ one part;
then, $10 - x =$ the other part;
and, $100 - x^2 = 96$;
whence, $x = 2$;
and, $10 + x = 12$, one part;
$10 - x = 8$, the other part.

13. Let $16 + x =$ one part;
then, $16 - x =$ the other part;
and, $256 - x^2 = 240$;
whence, $x = 4$;
and, $16 + x = 20$, one part;
$16 - x = 12$, the other part.

14. Let $x =$ the number of yards;
then, $\frac{3x}{8} =$ the price per yard;
and, $\frac{3x^2}{8} = 24$;
whence, $x = 8$, the number of yards.

15. Let $x =$ the breadth;
then, $\frac{10x}{9} =$ the length;
and, $\frac{10x^2}{9} = 1440$ rods;
whence, $x = 36$ rods, the breadth;
and, $\frac{10x}{9} = 40$ rods, the length.

16. Let $5x =$ one number;
then, $4x =$ the other;
and, $25x^2 + 16x^2 = 164$;
whence, $x = 2$;
and, $\left.\begin{array}{l} 5x = 10 \\ 4x = 8 \end{array}\right\}$ the numbers.

17. Let $3x =$ the first number;
$5x =$ the second number;
$8x =$ the third number;
then, $9x^2 + 25x^2 + 64x^2 = 392$;
whence, $x = 2$;
and,
$$\left.\begin{aligned}3x &= 6\\ 5x &= 10\\ 8x &= 16\end{aligned}\right\}\text{the numbers.}$$

18. Let $x =$ sum received per day;
then, $10x =$ days he worked;
and, $10x^2 = \$62.50$;
whence, $x = \$2.50$, sum received per day;
and, $10x = 25$, days he worked.

19. Let $x =$ the greater;
then, $\frac{324}{x} =$ the less;
and, $\frac{x^2}{324} = 4$;
whence, $x^2 = 1296$;
and,
$$\left.\begin{aligned}x &= 36, \text{ greater};\\ \frac{324}{x} &= 9, \text{ less};\end{aligned}\right\}\text{the numbers.}$$

20. Let $10 + x =$ side of larger in rods;
and, $10 - x =$ side of smaller in rods;
then, $100 + 20x + x^2 + 100 - 20x + x^2 = 272$;
whence, $2x^2 = 72$;
and, $x = 6$;
whence, $10 + x = 16$ rods, side of greater lot;
and, $10 - x = 4$ rods, side of smaller lot.

21. Let $x =$ one;
and, $\sqrt{170 - x^2} =$ the other;
then, $170 - 2x^2 = 72$;
whence, $x = 7$;
and, $\sqrt{170 - x^2} = 11$.

AFFECTED QUADRATICS.

Page 235.

56. Clearing, $9cx^2 + 8cx = 20c.$

Dividing by c, $9x^2 + 8x = 20.$

Completing square, $81x^2 + 72x + 16 = 196.$

Extracting square root, $9x + 4 = \pm 14;$

whence, $9x = 10, \text{ or } -18;$

and, $x = \frac{10}{9}, \text{ or } -2.$

57. Completing square,

$$x^2 + 2(a+8)x + (a+8)^2 = (a-8)^2.$$

Extracting square root, $x + a + 8 = \pm(a - 8);$

whence, $x = -16, \text{ or } -2a.$

58. Clearing, $3bx - 3x^2 + bx + x^2 = 15b + 15x.$

Transposing, etc., $2x^2 + (15 - 4b)x = -15b.$

Completing square,

$$16x^2 + 8(15 - 4b)x + (15 - 4b)^2 = 225 - 240b + 16b^2.$$

Extracting square root, $4x + 15 - 4b = \pm\sqrt{225 - 240b + 16b^2};$

whence, $x = \frac{4b - 15}{4} \pm \frac{1}{4}\sqrt{225 - 240b + 16b^2}.$

59. Clearing, $abx^2 + (a^2 - b^2)x = ab.$

Completing square,

$$4a^2b^2x^2 + 4ab(a^2 - b^2)x + (a^2 - b^2)^2 = (a^2 + b^2)^2.$$

Extracting square root, $2abx + a^2 - b^2 = \pm(a^2 + b^2);$

whence, $2abx = 2b^2, \text{ or } -2a^2;$

and, $x = \frac{b}{a}, \text{ or } -\frac{a}{b}.$

60. Extracting the square root of each member,

$$\frac{c}{x+c} = \pm\frac{d}{x-c}.$$

Clearing, $cx - c^2 = dx + cd, \text{ or } -dx - cd;$

whence, $x = \frac{c(c+d)}{c-d}, \text{ or } \frac{c(c-d)}{c+d}.$

61. Clearing, $8x^2 - 10bx = -3b^2.$

Completing square, $64x^2 - 80bx + 25b^2 = b^2.$

Extracting square root, $8x - 5b = \pm b;$

whence, $x = \frac{3b}{4}$, or $\frac{b}{2}.$

62. Clearing and transposing, $4x^2 - 15cx = 4c^2.$

Completing square, $64x^2 - 240cx + 225c^2 = 289c^2.$

Extracting square root, $8x - 15c = \pm 17c;$

whence, $x = 4c$, or $-\frac{c}{4}.$

63. Clearing and transposing,

$$a^2c^2x^2 - 2ab^2cx = -b^4;$$

whence, $x^2 - 2\frac{b^2}{ac}x = -\frac{b^4}{a^2c^2}.$

Completing square, $x^2 - 2\frac{b^2}{ac}x + \frac{b^4}{a^2c^2} = 0.$

Extracting square root, $x - \frac{b^2}{ac} = 0;$

whence, $x = \frac{b^2}{ac}.$

64. Clearing and transposing, $x\sqrt{c} - (c+1)\sqrt{x} = -\sqrt{c}.$

Completing square, $4cx - 4\sqrt{c}(c+1)\sqrt{x} + (c+1)^2 = (c-1)^2.$

Extracting square root, $2\sqrt{cx} - (c+1) = \pm(c-1);$

whence, $2\sqrt{cx} = 2c$, or $2;$

and, $x = c$, or $\frac{1}{c}.$

65. Clearing and transposing, $28x^2 - 24cx = 4c^2.$

Dividing by 4, and completing square, $49x^2 - 42cx + 9c^2 = 16c^2.$

Extracting square root, $7x - 3c = \pm 4c;$

whence, $x = c$, or $-\frac{c}{7}.$

66. Clearing, $a^2x^2 - 5abx = -6b^2.$

Completing square, $a^2x^2 - 5abx + \frac{25b^2}{4} = \frac{b^2}{4}.$

Extracting square root, $ax - \frac{5b}{2} = \pm\frac{b}{2};$

whence, $x = \frac{3b}{a}$, or $\frac{2b}{a}.$

67. Transposing, $cx^2-(c^2+1)x=-c.$

Completing square,

$$4c^2x^2-4c(c^2+1)x+(c^2+1)^2=(c^2-1)^2.$$

Extracting square root, $2cx-(c^2+1)=\pm(c^2-1);$

whence, $2cx=2c^2$, or $2;$

and, $x=c$, or $\frac{1}{c}.$

68. Clearing and transposing, $x^2-2x=-3.$

Completing square, $x^2-2x+1=-2.$

Extracting square root, $x-1=\pm\sqrt{-2};$

whence, $x=1\pm\sqrt{-2}.$

69. Completing square,

$$36x^2-12(b+2c)x+(b+2c)^2=(b-2c)^2.$$

Extracting square root, $6x-(b+2c)=\pm(b-2c);$

whence, $6x=2b$, or $4c;$

and, $x=\frac{b}{3}$, or $\frac{2c}{3}.$

70. Transposing and completing square,

$$x^2-(a-b+c)x+\frac{(a-b+c)^2}{4}=\frac{(a-b-c)^2}{4}.$$

Extracting square root, $x-\left(\frac{a-b+c}{2}\right)=\pm\left(\frac{a-b-c}{2}\right);$

whence, $x=a-b$, or $c.$

71. Squaring and transposing,

$$x^2-(a+5c)x=-4ac-4c^2.$$

Completing square,

$$x^2-(a+5c)x+\frac{(a+5c)^2}{4}=\frac{(a-3c)^2}{4}.$$

Extracting square root, $x-\frac{a+5c}{2}=\pm\frac{a-3c}{2};$

whence, $x=a+c$, or $4c.$

72. Completing square,

$$4(c+d)^2x^2-4(c^2-d^2)x+(c-d)^2=(c+d)^2.$$

Extracting square root, $2(c+d)x-(c-d)=\pm(c+d);$

whence, $x=\frac{c}{c+d}$, or $\frac{-d}{c+d}.$

73. Clearing and transposing,

$$3\sqrt{2x^2 - 2nx} = 9n + 3x.$$

Squaring both members, $18x^2 - 18nx = 81n^2 + 54nx + 9x^2.$

Transposing and dividing by 9, $x^2 - 8nx = 9n^2.$

Completing square, $x^2 - 8nx + 16n^2 = 25n^2.$

Extracting square root, $x - 4n = \pm 5n;$

whence, $x = 9n$, or $-n.$

74. Clearing and transposing,

$$(ac + bc)x^2 - [(a+b)^2 + c^2]x = -(ac + bc).$$

Completing square,

$$(ac+bc)^2x^2 - (ac+bc)[(a+b)^2+c^2]x + \frac{[(a+b)^2+c^2]^2}{4} = \frac{[(a+b)^2-c^2]^2}{4}.$$

Extracting square root,

$$c(a+b)x - \frac{[(a+b)^2+c^2]}{2} = \pm\frac{[(a+b)^2-c^2]}{2};$$

whence, $c(a+b)x = (a+b)^2$, or $c^2;$

and, $x = \frac{a+b}{c}$, or $\frac{c}{a+b}.$

75. Clearing and uniting terms,

$$(3a - 2b)x^2 + 6(b^2 - a^2)x = 12ab^2 - 8a^2b.$$

Completing square,

$$(3a-2b)^2x^2 + 6(3a-2b)(b^2-a^2)x + 9(b^2-a^2)^2 = (3b^2 - 4ab + 3a^2)^2.$$

Extracting square root,

$$(3a - 2b)x + 3(b^2 - a^2) = \pm(3b^2 - 4ab + 3a^2);$$

whence, $(3a - 2b)x = 6a^2 - 4ab$, or $4ab - 6b^2;$

and, $x = 2a$, or $\frac{2b(2a - 3b)}{3a - 2b}.$

76. Expanding and collecting terms,

$$2(c^2 - d^2)x^2 - 4(c^2 + d^2)x = 2(d^2 - c^2).$$

Dividing by 2, and completing square,

$$(c^2 - d^2)^2x^2 - 2(c^2 - d^2)(c^2 + d^2)x + (c^2 + d^2)^2 = 4c^2d^2.$$

Extracting square root, $(c^2 - d^2)x - (c^2 + d^2) = \pm 2cd;$

whence, $(c^2 - d^2)x = c^2 \pm 2cd + d^2;$

and, $x = \frac{c+d}{c-d}$, or $\frac{c-d}{c+d}.$

EQUATIONS IN THE QUADRATIC FORM.

Page 237.

4. Completing square, $x^4 - 2x^2 + 1 = 9.$
Extracting square root, $x^2 - 1 = \pm 3;$
whence, $x^2 = 4, \text{ or } -2;$
and, $x = \pm 2, \text{ or } \pm\sqrt{-2}.$

5. Completing square, $x^6 - 3x^3 + \frac{9}{4} = \frac{169}{4}.$
Extracting square root, $x^3 - \frac{3}{2} = \pm\frac{13}{2};$
whence, $x^3 = 8, \text{ or } -5;$
and, $x = 2, \text{ or } \sqrt[3]{-5}.$

6. Completing square, $x^6 - 4x^3 + 4 = 36.$
Extracting square root, $x^3 - 2 = \pm 6;$
whence, $x^3 = 8, \text{ or } -4;$
and, $x = 2, \text{ or } \sqrt[3]{-4}.$

7. Completing square, $x^4 - 2x^2 + 1 = 9.$
Extracting square root, $x^2 - 1 = \pm 3;$
whence, $x^2 = 4, \text{ or } -2;$
and, $x = \pm 2, \text{ or } \pm\sqrt{-2}.$

8. Completing square, $16x^4 - 4x^2 + \frac{1}{4} = \frac{49}{4}.$
Extracting square root, $4x^2 - \frac{1}{2} = \pm\frac{7}{2};$
whence, $4x^2 = 4, \text{ or } -3;$
and, $x = \pm 1, \text{ or } \pm\frac{1}{2}\sqrt{-3}.$

9. Dividing by 2, and completing square,

$$x^6 - \frac{x^3}{4} + \frac{1}{64} = \frac{3969}{64}.$$

Extracting square root, $x^3 - \frac{1}{8} = \pm\frac{63}{8};$
whence, $x^3 = 8, \text{ or } -\frac{31}{4};$
and, $x = 2, \text{ or } \sqrt[3]{-7\frac{3}{4}} = \frac{1}{2}\sqrt[3]{-62}.$

10. Completing square, $x^4 + 4x^2 + 4 = 16.$
Extracting square root, $x^2 + 2 = \pm 4;$
whence, $x^2 = 2, \text{ or } -6;$
and, $x = \pm\sqrt{2}, \text{ or } \pm\sqrt{-6}.$

11. Completing square, $x^{\frac{1}{2}} + 2x^{\frac{1}{4}} + 1 = 9.$

Extracting square root, $x^{\frac{1}{4}} + 1 = \pm 3;$

whence, $x^{\frac{1}{4}} = 2, \text{ or } -4.$

Raising to 4th power, $x = 16, \text{ or } 256.$

12. Completing square, $x^3 + 3x^{\frac{3}{2}} + \frac{9}{4} = \frac{361}{4}.$

Extracting square root, $x^{\frac{3}{2}} + \frac{3}{2} = \pm\frac{19}{2};$

whence, $x^{\frac{3}{2}} = 8, \text{ or } -11.$

Squaring, $x^3 = 64, \text{ or } 121;$

whence, $x = 4, \text{ or } \sqrt[3]{121}.$

13. Completing square, $x^{\frac{2}{3}} + 3x^{\frac{1}{3}} + \frac{9}{4} = \frac{25}{4}.$

Extracting square root, $x^{\frac{1}{3}} + \frac{3}{2} = \pm\frac{5}{2};$

whence, $x^{\frac{1}{3}} = 1, \text{ or } -4.$

Cubing, $x = 1, \text{ or } -64.$

14. Completing square, $x^{\frac{1}{2}} - x^{\frac{1}{4}} + \frac{1}{4} = \frac{81}{4}.$

Extracting square root, $x^{\frac{1}{4}} - \frac{1}{2} = \pm\frac{9}{2};$

whence, $x^{\frac{1}{4}} = -4, \text{ or } 5.$

Raising to 4th power, $x = 256, \text{ or } 625.$

15. Completing square,

$$4a^2x^{2n} + 4abx^n + b^2 = 4ac + b^2.$$

Extracting square root, $2ax^n + b = \pm\sqrt{4ac + b^2};$

whence, $x^n = -\frac{b}{2a} \pm \frac{1}{2a}\sqrt{4ac + b^2}.$

Extracting *n*th root, $x = \sqrt[n]{-\frac{b}{2a} \pm \frac{1}{2a}\sqrt{4ac + b^2}}.$

16. Completing square,

$$(x^2 + 4)^2 + (x^2 + 4) + \frac{1}{4} = \frac{121}{4}.$$

Extracting square root, $x^2 + 4 + \frac{1}{2} = \pm\frac{11}{2};$

whence, $x^2 = 1, \text{ or } -10;$

and, $x = \pm 1, \text{ or } \pm\sqrt{-10}.$

17. Completing square,

$$4(x-1)^2+20(x-1)+25=81.$$

Extracting square root, $2(x-1)+5=\pm 9;$

whence, $2(x-1)=4, \text{ or } -14;$

and, $x=3, \text{ or } -6.$

18. Completing square,

$$4(x^2-9)^2-44(x^2-9)+121=441.$$

Extracting square root, $2(x^2-9)-11=\pm 21;$

whence, $2(x^2-9)=32, \text{ or } -10;$

and, $x^2=25, \text{ or } 4;$

therefore, $x=\pm 5, \text{ or } \pm 2.$

19. Completing square,

$$(x^2+1)^2+(x^2+1)+\tfrac{1}{4}=\tfrac{121}{4}.$$

Extracting square root, $x^2+1+\tfrac{1}{2}=\pm\tfrac{11}{2};$

whence, $x^2=4, \text{ or } -7;$

and, $x=\pm 2, \text{ or } \pm\sqrt{-7}.$

20. Completing square,

$$(x^2-x)^2-(x^2-x)+\tfrac{1}{4}=\tfrac{529}{4}.$$

Extracting square root, $x^2-x+\tfrac{1}{2}=\pm\tfrac{23}{2};$

whence, $x^2-x=12, \text{ or } -11.$

Completing square, $x^2-x+\tfrac{1}{4}=\tfrac{49}{4}, \text{ or } -\tfrac{43}{4}.$

Extracting square root, $x-\tfrac{1}{2}=\pm\tfrac{7}{2}, \text{ or } \pm\tfrac{1}{2}\sqrt{-43};$

whence, $x=4, \text{ or } -3, \text{ or } \tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{-43}.$

21. Completing square,

$$(x+5)-\sqrt{x+5}+\tfrac{1}{4}=\tfrac{25}{4}.$$

Extracting square root, $\sqrt{x+5}+\tfrac{1}{2}=\pm\tfrac{5}{2};$

whence, $\sqrt{x+5}=2, \text{ or } -3.$

Squaring, $x+5=4, \text{ or } 9;$

therefore, $x=-1, \text{ or } 4.$

22. Completing square,

$$(3x+4)+4\sqrt{3x+4}+4=36.$$

Extracting square root, $\sqrt{3x+4}+2=\pm 6;$

whence, $\sqrt{3x+4}=4, \text{ or } -8.$

Squaring, etc., $x=4, \text{ or } 20.$

23. Completing square,

$$\left(\frac{8}{x}+x\right)^2+\left(\frac{8}{x}+x\right)+\frac{1}{4}=\frac{169}{4}.$$

Extracting square root, $\frac{8}{x}+x+\frac{1}{2}=\pm\frac{13}{2}$;

whence, $\frac{8}{x}+x=6$, or -7.

Clearing, etc., $x^2-6x=-8$, or $x^2+7x=-8$.

Completing square, $4x^2-24x+36=4$, or $4x^2+28x+49=17$.

Extracting square root, $2x-6=\pm 2$, or $2x+7=\pm\sqrt{17}$;

whence, $x=4$, or 2, or $x=-\frac{7}{2}\pm\frac{1}{2}\sqrt{17}$.

24. Completing square,

$$\left(\frac{6}{x}+x\right)^2+\left(\frac{6}{x}+x\right)+\frac{1}{4}=\frac{121}{4}.$$

Extracting square root, $\frac{6}{x}+x+\frac{1}{2}=\pm\frac{11}{2}$;

whence, $\frac{6}{x}+x=5$, or -6.

Clearing, etc., $x^2-5x=-6$, or $x^2+6x=-6$.

Completing square, $4x^2-20x+25=1$, or $4x^2+24x+36=12$.

Extracting square root, $2x-5=\pm 1$, or $2x+6=\pm\sqrt{12}$;

whence, $x=3$ or 2, or $x=-3\pm\sqrt{3}$.

25. Completing square,

$$(2x^2-4x+1)^2-(2x^2-4x+1)+\tfrac{1}{4}=\tfrac{169}{4}.$$

Extracting square root, $(2x^2-4x+1)-\frac{1}{2}=\pm\frac{13}{2}$.

Transposing, etc., $2x^2-4x=6$, or -7.

Completing square, $4x^2-8x+4=16$, or -10.

Extracting square root, $2x-2=\pm 4$, or $\pm\sqrt{-10}$;

whence, $x=3,\ -1$, or $1\pm\frac{1}{2}\sqrt{-10}$.

26. Completing square,

$$x^2-7x+18+\sqrt{x^2-7x+18}+\tfrac{1}{4}=\tfrac{169}{4}.$$

Extracting square root, $\sqrt{x^2-7x+18}+\frac{1}{2}=\pm\frac{13}{2}$.

Transposing, etc., $\sqrt{x^2-7x+18}=6$, or -7.

Squaring, $x^2-7x+18=36$, or 49.

Transposing, etc., $x^2-7x=18$, or 31.

Completing square, $4x^2-28x+49=121$, or 173.

Extracting square root, $2x-7=\pm 11$, or $\pm\sqrt{173}$.

Therefore, $x=9,\ -2$, or $\frac{7}{2}\pm\frac{1}{2}\sqrt{173}$.

27. Completing square,

$$2x^2+3x+9-5\sqrt{2x^2+3x+9}+\tfrac{25}{4}=\tfrac{49}{4}.$$

Extracting square root, $\sqrt{2x^2+3x+9}-\tfrac{5}{2}=\pm\tfrac{7}{2}.$

Transposing, etc., $\sqrt{2x^2+3x+9}=-1$, or $6.$

Squaring, $2x^2+3x+9=1$, or $36.$

Transposing, etc., $2x^2+3x=-8$, or $27.$

Completing square, $16x^2+24x+9=-55$, or $225.$

Extracting square root, $4x+3=\pm\sqrt{-55}$, or $\pm 15;$

whence, $x=-\tfrac{3}{4}\pm\tfrac{1}{4}\sqrt{-55}$, 3, or $-4\tfrac{1}{2}$

Page 238.

28. Completing square,

$$3x^2+1+2(3x^2+1)^{\frac{1}{2}}+1=64.$$

Extracting square root, $\sqrt{3x^2+1}+1=\pm 8.$

Transposing, etc., $\sqrt{3x^2+1}=7$, or $-9.$

Squaring, $3x^2+1=49$, or $81.$

whence, $x=\pm 4$, or $\pm\sqrt{\tfrac{80}{3}}$, $=\pm\tfrac{4}{3}\sqrt{15}.$

29. Completing square,

$$\sqrt{x+12}+\sqrt[4]{x+12}+\tfrac{1}{4}=\tfrac{25}{4}.$$

Extracting square root, $\sqrt[4]{x+12}+\tfrac{1}{2}=\pm\tfrac{5}{2}.$

Transposing, etc., $\sqrt{x+12}=2$, or $-3.$

Raising to 4th power, $x+12=16$, or $81;$

whence, $x=4$, or $69.$

30. Completing square, $x+2+(x+2)^{\frac{1}{2}}+\tfrac{1}{4}=\tfrac{81}{4}.$

Extracting square root, $\sqrt{x+2}+\tfrac{1}{2}=\pm\tfrac{9}{2}.$

Transposing, etc., $\sqrt{x+2}=4$, or $-5.$

Squaring, $x+2=16$, or $25;$

whence, $x=14$, or $23.$

31. Completing square, $(x+4)^2+(x+4)+\tfrac{1}{4}=\tfrac{121}{4}.$

Extracting square root, $x+4+\tfrac{1}{2}=\pm\tfrac{11}{2};$

whence, $x+4=5$, or $-6;$

and, $x=1$, or $-10.$

32. Completing square,

$$\sqrt{x+31}-\tfrac{1}{3}\sqrt[4]{x+31}+\tfrac{1}{36}=\tfrac{289}{36}.$$

Extracting square root, $\sqrt[4]{x+31}-\tfrac{1}{6}=\pm\tfrac{17}{6};$

whence, $\sqrt[4]{x+31}=3$, or $-\tfrac{8}{3}.$

Raising each number to 4th power, $x+31=81$, or $\tfrac{4096}{81};$

whence, $x=50$, or $19\tfrac{46}{81}.$

33. Add 13 to each member.

Completing square,

$$4x^2 + 13 - 4\sqrt{4x^2+13} + 4 = 25.$$

Extracting square root, $\sqrt{4x^2+13} - 2 = \pm 5$;

whence, $\sqrt{4x^2+13} = 7$, or -3.

Squaring, $4x^2 + 13 = 49$, or 9;

whence, $x = \pm 3$, or $\pm\sqrt{-1}$.

34. Collecting terms, $x^2 + (2b+1)x = 6 - b - b^2$.

Completing square, $x^2 + (2b+1)x + \frac{(2b+1)^2}{4} = \frac{25}{4}$.

Extracting square root, $x + \frac{2b+1}{2} = \pm\frac{5}{2}$;

whence, $x = 2 - b$, or $-(3+b)$.

35. Add 8 to each member.

Completing square,

$$(x^2 + 4x + 4)^2 + 2(x^2 + 4x + 4) + 1 = 16.$$

Extracting square root, $x^2 + 4x + 4 + 1 = \pm 4$.

Transposing, etc., $x^2 + 4x = -1$, or -9.

Completing square again, $x^2 + 4x + 4 = 3$, or -5.

Extracting square root, $x + 2 = \pm\sqrt{3}$, or $\pm\sqrt{-5}$;

whence, $x = -2 \pm \sqrt{3}$, or $-2 \pm \sqrt{-5}$.

36. Completing square,

$$x^2 - 2x - 3\sqrt{x^2-2x} + \tfrac{9}{4} = \tfrac{25}{4}.$$

Extracting square root, $\sqrt{x^2-2x} - \frac{3}{2} = \pm\frac{5}{2}$;

whence, $\sqrt{x^2-2x} = 4$, or -1.

Squaring, $x^2 - 2x = 16$, or 1.

Completing square, $x^2 - 2x + 1 = 17$, or 2.

Extracting root, $x - 1 = \pm\sqrt{17}$, or $\pm\sqrt{2}$;

whence, $x = 1 \pm \sqrt{17}$, or $1 \pm \sqrt{2}$.

37. Completing square,

$$\sqrt{x+a} + 2b\sqrt[4]{x+a} + b^2 = 4b^2.$$

Extracting square root, $\sqrt[4]{x+a} + b = \pm 2b$;

whence, $\sqrt[4]{x+a} = b$, or $-3b$.

Raising to 4th power, $x + a = b^4$, or $81b^4$;

whence, $x = b^4 - a$, or $81b^4 - a$.

38. Add 18 to each member.

Completing square,

$$x^2 - 10x + 18 - 2\sqrt{x^2 - 10x + 18} + 1 = 4.$$

Extracting square root, $\sqrt{x^2 - 10x + 18} - 1 = \pm 2$;

whence, $\sqrt{x^2 - 10x + 18} = 3$, or -1.

Squaring, $x^2 - 10x + 18 = 9$, or 1.

Transposing and completing square,

$$x^2 - 10x + 25 = 16, \text{ or } 8.$$

Extracting square root, $x - 5 = \pm 4$, or $\pm 2\sqrt{2}$;

whence, $x = 9$ or 1, or $5 \pm 2\sqrt{2}$.

QUADRATIC EQUATIONS.—PROBLEMS.

2. Let $x =$ one number;

then, $10 - x =$ the other number;

and, $10x - x^2 = 21$;

whence, $x = 7$, one number;

and, $10 - x = 3$, the other number.

3. Let $x =$ one part;

then, $27 - x =$ the other part;

and, $27x - x^2 = 140$;

whence, $x = 7$, one part;

and, $27 - x = 20$, the other part.

Page 239.

4. Let $x =$ width in rods;

then, $x + 12 =$ length in rods;

and, $\dfrac{x^2 + 12x}{160} = 7$;

whence, $x = 28$ rods, the width;

and, $x + 12 = 40$ rods, the length.

5. Let $x =$ the number purchased;

then, $\dfrac{100}{x} =$ the cost of each;

and, $\dfrac{100}{x} - \dfrac{100}{x + 5} = 1$;

whence, $x = 20$ sheep.

6. Let $x =$ number trees in a row;

then, $x + 10 =$ number of rows;

and, $x^2 + 10x = 2000$;

whence, $x = 40$, number in a row;

and, $x + 10 = 50$, number of rows.

7. Let $x =$ one number;

then, $x + 2 =$ the other number;

and, $x^2 + x^2 + 4x + 4 = 244$;

whence, $x = 10$;

and, $x + 2 = 12$.

8. Let $x =$ number of persons;

then, $x^2 - 110 = x$;

whence, $x = 11$ persons.

9. Let $x =$ daily wages;

then, $\frac{18}{x} =$ days worked

and, $\frac{18}{x-1} - \frac{18}{x} = 3$;

whence, $x = 3$;

and, $\frac{18}{x} = 6$, number of days worked.

10. Let $x =$ number of eggs for 12 cents;

then, $\frac{12}{x} =$ price per egg.

and, $\frac{12}{x-2} - \frac{12}{x} = \frac{1}{12}$;

whence, $x = 18$ eggs for 12 cents; or 8 cents per dozen.

11. Let $x =$ the cost;

then, $\frac{x}{100} =$ gain per cent;

and, $\frac{x^2}{100} =$ entire gain.

Hence, $x + \frac{x^2}{100} = 24$;

whence, $x = 20$ dollars, the cost.

12. Let $x =$ number of persons;
then, $x + 30 =$ cents paid by each;
and, $x^2 + 30x = 1000$;
whence, $x = 20$ persons.

13. Let $x =$ width of frame;
then, $4x^2 + 60x =$ area of frame;
and, $4x^2 + 60x = 216$;
whence, $x = 3$ inches, the width.

14. Let $x =$ the cost;
then, $\frac{x}{100} =$ gain per cent;
and, $\frac{x^2}{100} =$ entire gain.
Hence, $x + \frac{x^2}{100} = 39$;
whence, $x = 30$ dollars, the cost.

Page 240.

15. Let $x =$ rods dug by A;
then, $100 - x =$ rods dug by B;
$\frac{50}{x} =$ A's price per rod;
$\frac{50}{100 - x} =$ B's price per rod.
Therefore, $\frac{50}{x} - \frac{50}{100 - x} = \frac{1}{4}$;
whence, $x = 43.845-$, rods dug by A;
and, $100 - x = 56.155+$, rods dug by B;
$\frac{\$50}{43.845} = \$1.14+$, A's price per rod;
$\frac{\$50}{56.155} = \$.89+$, B's price per rod.

16. Let $x =$ width of street;
then, $4x^2 + 200x =$ area of street;
and, $4x^2 + 200x = 1344$;
whence, $x = 6$ rods, the width.

17. Let $x =$ yards in one piece;
then, $36 - x =$ yards in the other piece;
$x^2 =$ cost of one piece;
$(36 - x)^2 =$ cost of the other piece.
Therefore, $4x^2 = (36 - x)^2$;
whence, $x = 12$ yards;
and, $36 - x = 24$ yards.

18. Let $x =$ quantity drawn each time;
then, $81 - x =$ quantity of pure wine remaining after first drawing;
and, $\frac{81 - x}{81} =$ part of each gallon of mixture which is pure wine;
$\frac{81x - x^2}{81} =$ quantity of *wine* drawn second time.
Hence, $x + \frac{81x - x^2}{81} = 17$;
whence, $x = 9$ gallons.

19. Let $x =$ B's rate per hour;
then, $x + 1 =$ A's rate per hour;
$\frac{90}{x} =$ B's time;
$\frac{90}{x + 1} =$ A's time.
Hence, $\frac{90}{x} - \frac{90}{x + 1} = 1$;
whence, $x = 9$ miles per hour;
and, $x + 1 = 10$ miles per hour.

20. Let $x =$ number of silver coins;
then, $27 - x =$ number of copper coins;
and, $27x - x^2 =$ entire value of silver or copper coins.
Therefore, $54x - 2x^2 = 100$;
whence, $x = 2$, the number of silver coins;
and, $27 - x = 25$, the number of copper coins.

SIMULTANEOUS QUADRATIC EQUATIONS.

Page 244.

7. $x - y = 4;$ (1)

$x^2 - y^2 = 32.$ (2)

Dividing (2) by (1), $x + y = 8.$ (3)

Adding (3) and (1), $2x = 12;$

whence, $x = 6;$

and, $y = 2.$

8. $x + y = 5;$ (1)

$x^2 + y^2 = 13.$ (2)

Squaring (1), $x^2 + 2xy + y^2 = 25.$ (3)

Subtracting (2) from (3), $2xy = 12.$ (4)

Subtracting (4) from (2), $x^2 - 2xy + y^2 = 1.$ (5)

Extracting square root, $x - y = \pm 1.$ (6)

Adding (1) and (6), $2x = 6, \text{ or } 4;$

whence, $x = 3, \text{ or } 2;$

and, $y = 2, \text{ or } 3.$

9. $xy = 15;$ (1)

$2x + y = 13.$ (2)

From (1), $y = \frac{15}{x}.$ (3)

Substituting in (2), $2x + \frac{15}{x} = 13.$ (4)

Clearing, etc., $x = 5, \text{ or } 1\frac{1}{2};$

and, $y = 3, \text{ or } 10.$

10. $xy = 6;$ (1)

$2x - 3y = 9.$ (2)

From (1), $x = \frac{6}{y}.$ (3)

Substituting in (2), $\frac{12}{y} - 3y = 9.$ (4)

Clearing, etc., $y = 1, \text{ or } -4;$

and, $x = 6, \text{ or } -1\frac{1}{2}.$

11. $x + 2y = 7;$ (1)

$2x^2 - y^2 = 14.$ (2)

From (1), $x = 7 - 2y.$ (3)

Substituting in (2), $98 - 56y + 8y^2 - y^2 = 14.$

Combining terms, etc., $y = 2, \text{ or } 6;$

$x = 3, \text{ or } -5.$

12. $2x + 3y = 22;$ (1)

$2xy = 40.$ (2)

From (2), $x = \frac{20}{y}.$ (3)

Substituting in (1), $\frac{40}{y} + 3y = 22.$ (4)

Clearing, etc., $y = 4, \text{ or } 3\frac{1}{3};$

and, $x = 5, \text{ or } 6.$

13. $\frac{1}{x} + \frac{1}{y} = \frac{3}{10};$ (1)

$\frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{20}.$ (2)

Squaring (1), $\frac{1}{x^2} + \frac{2}{xy} + \frac{1}{y^2} = \frac{9}{100}.$ (3)

Subtracting (2) from (3), $\frac{2}{xy} = \frac{4}{100}.$ (4)

Subtracting (4) from (2), $\frac{1}{x^2} - \frac{2}{xy} + \frac{1}{y^2} = \frac{1}{100}.$ (5)

Extracting the square root, $\frac{1}{x} - \frac{1}{y} = \pm\frac{1}{10}.$ (6)

Adding (6) and (1), $\frac{2}{x} = \frac{4}{10}, \text{ or } \frac{2}{10};$

whence, $x = 5, \text{ or } 10;$

and, $y = 10, \text{ or } 5.$

14. $xy = 24;$ (1)

$x^2 - 2y^2 = 4.$ (2)

From (1), $x = \frac{24}{y}.$ (3)

Substituting in (2), $\frac{576}{y^2} - 2y^2 = 4.$ (4)

Clearing, etc., $y = \pm 4, \text{ or } \mp 3\sqrt{-2};$

and, $x = \pm 6, \text{ or } \pm 4\sqrt{-2}.$

15. $xy = 10;$ (1)

$x - y = 3.$ (2)

Squaring (2), $x^2 - 2xy + y^2 = 9.$ (3)

Multiplying (1) by 4, $4xy = 40.$ (4)

Adding (3) and (4), $x^2 + 2xy + y^2 = 49.$ (5)

Extracting square root, $x + y = \pm 7.$ (6)

Adding (6) and (2), $2x = 10, \text{ or } -4;$

whence, $x = 5, \text{ or } -2;\ y = 2, \text{ or } -5.$

16. $xy = 12;$ (1)

$x + y = 7.$ (2)

Squaring (2), $x^2 + 2xy + y^2 = 49.$ (3)

Multiplying (1) by 4, $4xy = 48.$ (4)

Subtracting (4) from (3), $x^2 - 2xy + y^2 = 1.$ (5)

Extracting square root, $x - y = \pm 1.$ (6)

Adding (6) and (2), $2x = 8, \text{ or } 6;$

whence, $x = 4, \text{ or } 3;$

and, $y = 3, \text{ or } 4.$

17. $x - y = 1;$ (1)

$x^3 - y^3 = 37.$ (2)

Dividing (2) by (1), $x^2 + xy + y^2 = 37.$ (3)

Squaring (1), $x^2 - 2xy + y^2 = 1.$ (4)

Subtracting (4) from (3), $3xy = 36;$ (5)

$xy = 12.$ (6)

Adding (6) to (3), $x^2 + 2xy + y^2 = 49.$ (7)

Extracting square root, $x + y = \pm 7.$ (8)

Adding (8) and (1), etc., $x = 4, \text{ or } -3;\ y = 3, \text{ or } -4.$

18. $x + y = 4;$ (1)

$x^3 + y^3 = 28.$ (2)

Dividing (2) by (1), $x^2 - xy + y^2 = 7.$ (3)

Squaring (1), $x^2 + 2xy + y^2 = 16.$ (4)

Subtracting (3) from (4), $3xy = 9;$ (5)

$xy = 3.$ (6)

Subtracting (6) from (3), $x^2 - 2xy + y^2 = 4.$ (7)

Extracting square root, $x - y = \pm 2.$ (8)

Adding (8) and (1), etc., $x = 3, \text{ or } 1;\ y = 1, \text{ or } 3.$

19. $x^3 + y^3 = 28;$ (1)

$x^2y + xy^2 = 12.$ (2)

Multiplying (2) by 3, $3x^2y + 3xy^2 = 36.$ (3)

Adding (1) and (3), $x^3 + 3x^2y + 3xy^2 + y^3 = 64.$ (4)

Extracting cube root, $x + y = 4.$ (5)

Dividing (2) by (5), $xy = 3.$ (6)

Squaring (5), $x^2 + 2xy + y^2 = 16.$ (7)

Multiplying (6) by 4, $4xy = 12.$ (8)

Subtracting (8) from (7), $x^2 - 2xy + y^2 = 4.$ (9)

Extracting square root, $x - y = \pm 2.$ (10)

Adding (10) and (5), etc., $x = 3$, or 1; $y = 1$, or 3.

20. $x^3 - y^3 = 26;$ (1)

$-x^2y + xy^2 = -6.$ (2)

Multiplying (2) by (3), $-3x^2y + 3xy^2 = -18.$ (3)

Adding (3) and (1), $x^3 - 3x^2y + 3xy^2 - y^3 = 8.$ (4)

Extracting root, $x - y = 2.$ (5)

Dividing (2) by (5), $xy = 3.$ (6)

Dividing (1) by (5), $x^2 + xy + y^2 = 13.$ (7)

Adding (7) and (6), $x^2 + 2xy + y^2 = 16.$ (8)

Extracting square root, $x + y = \pm 4.$ (9)

Adding (5) and (9), etc., $x = 3$, or -1; $y = 1$, or -3

21. $x^2 - xy = 6;$ (1)

$x^2 + y^2 = 61.$ (2)

Let $x = vy;$

then (1) becomes, $v^2y^2 - vy^2 = 6;$ (3)

and (2) becomes, $v^2y^2 + y^2 = 61.$ (4)

From (3), $y^2 = \frac{6}{v^2 - v}.$ (5)

From (4), $y^2 = \frac{61}{v^2 + 1}.$ (6)

Equating (5) and (6), $\frac{6}{v^2 - v} = \frac{61}{v^2 + 1};$ (7)

whence, $v = \frac{6}{5}$, or $-\frac{1}{11}$. (8)

Substituting value of v in (5), $y = \pm 5$, or $\pm \frac{11}{2}\sqrt{2}$.

Therefore, $x = \pm 6$, or $\mp \frac{1}{2}\sqrt{2}$.

22. $3x^2 + xy = 18;$ (1)

$4y^2 + 3xy = 54.$ (2)

Let $x = vy;$

then (1) becomes, $3v^2y^2 + vy^2 = 18;$ (3)

and (2) becomes, $4y^2 + 3vy^2 = 54.$ (4)

Dividing (4) by (3) $\dfrac{4 + 3v}{3v^2 + v} = 3;$ (5)

whence, $v = \frac{2}{3}$, or $-\frac{2}{3}$. (6)

Substituting in (3), etc., $y = \pm 3$, or $\pm 3\sqrt{3}$; $x = \pm 2$, or $\mp 2\sqrt{3}$.

23. $x^2 + xy = 70;$ (1)

$xy - y^2 = 12.$ (2)

Let $x = vy;$

then (1) becomes, $v^2y^2 + vy^2 = 70;$ (3)

and (2) becomes, $vy^2 - y^2 = 12.$ (4)

From (3), $y^2 = \dfrac{70}{v^2 + v}.$ (5)

From (4), $y^2 = \dfrac{12}{v - 1}.$ (6)

Equating (5) and (6), $\dfrac{70}{v^2 + v} = \dfrac{12}{v - 1};$ (7)

whence, $v = \frac{7}{3}$, or $\frac{5}{2}$. (8)

Substituting in (6), etc., $y = \pm 3$, or $\pm 2\sqrt{2}$; $x = \pm 7$, or $\pm 5\sqrt{2}$.

24. $2x + y = 22;$ (1)

$xy + 2y^2 = 120.$ (2)

From (1), $x = \dfrac{22 - y}{2}.$ (3)

Substituting in (2), $\dfrac{22y - y^2}{2} + 2y^2 = 120;$ (4)

whence, $y = 6$, or $-13\frac{1}{3}$;

$x = 8$, or $17\frac{2}{3}$.

25. $x = 2y^2;$ (1)

$x - y = 15.$ (2)

Substituting (1) in (2), $2y^2 - y = 15;$ (3)

whence, $y = 3$, or $-2\frac{1}{2}$;

$x = 18$, or $12\frac{1}{2}$.

26. $x + 4y = 14;$ (1)

$y^2 - 2y + 4x = 11.$ (2)

From (1), $x = 14 - 4y.$ (3)

Substituting in (2), $y^2 - 2y + 56 - 16y = 11;$ (4)

whence, $y = 3, \text{ or } 15; \; x = 2, \text{ or } -46.$

27. $4xy + x^2y^2 = 96;$ (1)

$x + y = 6.$ (2)

Completing square in (1), $x^2y^2 + 4xy + 4 = 100.$ (3)

Extracting square root, $xy + 2 = \pm 10;$ (4)

whence, $xy = 8, \text{ or } -12.$ (5)

Combining (2) and (5), $x = 4, \text{ or } 2, \text{ or } 3 \pm \sqrt{21};$

$y = 2, \text{ or } 4, \text{ or } 3 \mp \sqrt{21}.$

Page 245.

28. $x^2 + y^2 = 52;$ (1)

$x + y + xy = 34.$ (2)

Multiplying (2) by 2, $2(x + y) + 2xy = 68.$ (3)

Adding (1) and (3), $x^2 + 2xy + y^2 + 2(x + y) = 120.$ (4)

Completing square, $(x + y)^2 + 2(x + y) + 1 = 121.$ (5)

Extracting root, etc., $x + y = 10, \text{ or } -12.$ (6)

Substituting in (2), $xy = 24, \text{ or } 46.$ (7)

Multiplying (7) by 2, $2xy = 48, \text{ or } 92.$ (8)

Subtracting (8) from (1), and extracting root,

$x - y = \pm 2, \text{ or } \pm 2\sqrt{-10}.$ (9)

Combining (6) and (9), $x = 6, \text{ or } 4, \text{ or } -6 \pm \sqrt{-10};$

$y = 4, \text{ or } 6, \text{ or } -6 \mp \sqrt{-10}.$

29. $x^2 + y^2 = 13;$ (1)

$xy + y^2 = 15.$ (2)

Let $x = vy;$

then (1) becomes, $v^2y^2 + y^2 = 13;$ (3)

and (2) becomes, $vy^2 + y^2 = 15.$ (4)

From (3), $y^2 = \dfrac{13}{v^2 + 1}.$ (5)

From (4), $y^2 = \dfrac{15}{v + 1}.$ (6)

Equating (5) and (6), $\frac{13}{v^2+1}=\frac{15}{v+1}$; (7)

whence, $v=\frac{2}{3}$, or $\frac{1}{5}$.

Substituting in (5), etc., $y=\pm 3$, or $\pm\frac{5}{2}\sqrt{2}$;

$x=\pm 2$, or $\pm\frac{1}{2}\sqrt{2}$.

30. $\sqrt[3]{x}+\sqrt[3]{y}=6$; (1)

$x+y=72$. (2)

Let $\sqrt[3]{x}=p$, and $\sqrt[3]{y}=q$;

then (1) becomes, $p+q=6$; (3)

and (2) becomes, $p^3+q^3=72$. (4)

Solving as in Ex. 18, $p=4$, or 2; and $q=2$, or 4;

whence, $x=64$, or 8; and $y=8$, or 64.

31. $x^3-y^3=56$; (1)

$x-y=\frac{16}{xy}$. (2)

Clearing (2), and multiplying by 3,

$3x^2y-3xy^2=48$. (3)

Subtracting (3) from (1),

$x^3-3x^2y+3xy^2-y^3=8$. (4)

Extracting cube root, $x-y=2$. (5)

Substituting in (2), $xy=8$. (6)

Solving, as in Ex. 15, $x=4$, or -2; $y=2$, or -4.

32. $x+y=3$; (1)

$x^4+y^4=17$. (2)

$(1)^4$, $x^4+4x^3y+6x^2y^2+4xy^3+y^4=81$. (3)

Subtracting (2) from (3),

$4x^3y+6x^2y^2+4xy^3=64$. (4)

Squaring (1), and multiplying by $4xy$,

$4x^3y+8x^2y^2+4xy^3=36xy$. (5)

Subtracting (4) from (5), $2x^2y^2=36xy-64$.

Completing square, etc., $xy=2$, or 16. (6)

Combining (1) and (6), as in Ex. 16, $x=2$, or 1, or $\frac{1}{2}(3\pm\sqrt{-55})$;

$y=1$, or 2, or $\frac{1}{2}(3\mp\sqrt{-55})$.

33.
$$(x-y)(x^2+y^2)=13; \qquad (1)$$
$$x^2y-xy^2=6. \qquad (2)$$

Multiplying (2) by 2, and factoring,
$$2xy(x-y)=12. \qquad (3)$$

Subtracting (3) from (1),
$$(x-y)(x^2-2xy+y^2)=1. \qquad (4)$$

Extracting cube root, $x-y=1.$ (5)

Dividing (2) by (5), $xy=6.$ (6)

Solving, as in Ex. 15, $x=3$, or -2; $y=2$, or -3.

34.
$$x^2+x+y=18-y^2; \qquad (1)$$
$$xy=6. \qquad (2)$$

Transposing in (1), $x^2+y^2+x+y=18.$ (3)

Multiplying (2) by 2, $2xy=12.$ (4)

Adding (4) to (3), $(x+y)^2+(x+y)=30.$

Completing square, etc., $x+y=5$, or -6.

Solving, as in Ex. 16, $x=3, 2$, or $-3\pm\sqrt{3}$; $y=2, 3$, or $-3\mp\sqrt{3}$.

35.
$$x^2-xy+y^2=21; \qquad (1)$$
$$y^2-2xy+15=0. \qquad (2)$$

Let $x=vy;$

then (1) becomes, $v^2y^2-vy^2+y^2=21;$ (3)

and (2) becomes, $y^2-2vy^2=-15.$ (4)

From (3),
$$y^2=\frac{21}{v^2-v+1}. \qquad (5)$$

From (4),
$$y^2=\frac{-15}{1-2v}. \qquad (6)$$

Equating (5) and (6),
$$\frac{21}{v^2-v+1}=\frac{-15}{1-2v}; \qquad (7)$$

whence, $v=\frac{4}{5}$, or 3.

Substituting in (5), $y=\pm 5$, or $\pm\sqrt{3}$.

Substituting in (2), $x=\pm 4$, or $\pm 3\sqrt{3}$.

36.
$$x^2+xy+y^2=39; \qquad (1)$$
$$2x^2+3xy+y^2=63. \qquad (2)$$

Let $x=vy;$

then (1) becomes, $v^2y^2+vy^2+y^2=39;$ (3)

and (2) becomes, $2v^2y^2+3vy^2+y^2=63.$ (4)

Equating (3) and (4), $\frac{39}{v^2+v+1}=\frac{63}{2v^2+3v+1}$; (5)

whence, $v=\frac{2}{3}$, or -4.

Substituting in (3), $y=\pm 5$, or $\pm\sqrt{3}$.

Substituting in (1), $x=\pm 2$, or $\mp 4\sqrt{3}$.

37. $x^2+xy+2y^2=74$; (1)

$2x^2+2xy+y^2=73$. (2)

Let $x=vy$;

then (1) becomes, $v^2y^2+vy^2+2y^2=74$; (3)

then (2) becomes, $2v^2y^2+2vy^2+y^2=73$. (4)

From (3), $y^2=\frac{74}{v^2+v+2}$. (5)

From (4), $y^2=\frac{73}{2v^2+2v+1}$. (6)

Equating (5) and (6), $\frac{74}{v^2+v+2}=\frac{73}{2v^2+2v+1}$; (7)

whence, $v=\frac{3}{5}$, or $-\frac{8}{5}$.

Substituting in (5), $y=\pm 5$.

Substituting in (1), $x=\pm 3$, or ∓ 8.

38. $x^2+y^2+xy=49$; (1)

$x^4+y^4+x^2y^2=931$. (2)

Dividing (2) by (1), $x^2-xy+y^2=19$. (3)

Subtracting (3) from (1), $2xy=30$. (4)

Adding (1) and (3), $x^2+y^2=34$. (5)

Adding (5) and (4), $x^2+2xy+y^2=64$. (6)

Extracting square root, $x+y=\pm 8$. (7)

Subtracting (4) from (5), $x^2-2xy+y^2=4$. (8)

Extracting square root, $x-y=\pm 2$. (9)

From (7) and (9), $x=\pm 5$, or ± 3;

$y=\pm 3$, or ± 5.

39. $x^4-x^2+y^4-y^2=84$; (1)

$x^2+x^2y^2+y^2=49$. (2)

Multiplying (2) by 2, and adding to (1),

$(x^2+y^2)^2+x^2+y^2=182$. (3)

Completing square, $(x^2+y^2)^2+(x^2+y^2)+\frac{1}{4}=\frac{729}{4}$. (4)

Extracting square root, $x^2 + y^2 + \frac{1}{2} = \pm \frac{27}{2}$; (5)
whence, $x^2 + y^2 = 13$, or -14. (6)
Substituting in (2), $x^2y^2 = 36$, or 63;
whence, $y^2 = \frac{36}{x^2}$, or $\frac{63}{x^2}$.
Substituting in (6), $x^2 + \frac{36}{x^2} = 13$, or $x^2 + \frac{63}{x^2} = -14$.
Solving, $x = 3,\ -3,\ 2,\ -2,\ \sqrt{-7 \pm \sqrt{-14}},\ -\sqrt{-7 \pm \sqrt{-14}}$;
$y = \pm 2,\ \pm 2,\ \pm 3,\ \pm 3,\ \pm\sqrt{-7 \mp \sqrt{-14}},\ \pm\sqrt{-7 \mp \sqrt{-14}}$.

40. $x + \sqrt{xy} + y = 19$; (1)
$x^2 + xy + y^2 = 133$. (2)
Dividing (2) by (1), $x - \sqrt{xy} + y = 7$. (3)
Adding (1) and (3), etc., $x + y = 13$. (4)
Subtracting (3) from (1), etc., $\sqrt{xy} = 6$. (5)
Squaring, $xy = 36$. (6)
Solving, as in Ex. 16, $x = 9$, or 4; $y = 4$, or 9.

41. $x^2 + 2y^2 = 44 - xy$; (1)
$2x^2 - 16 = 3xy - 2y^2$. (2)
Let $x = vy$;
then (1) becomes, $v^2y^2 + 2y^2 = 44 - vy^2$; (3)
and (2) becomes, $2v^2y^2 - 16 = 3vy^2 - 2y^2$. (4)
From (3), $y^2 = \frac{44}{v^2 + v + 2}$. (5)
From (4), $y^2 = \frac{16}{2v^2 - 3v + 2}$. (6)
Equating (5) and (6), $\frac{11}{v^2 + v + 2} = \frac{4}{2v^2 - 3v + 2}$; (7)
whence, $v = \frac{1}{2}$, or $\frac{14}{9}$.
Substituting in (5), $y = \pm 4$, or $\pm \frac{9}{11}\sqrt{11}$.
Substituting in (1), $x = \pm 2$, or $\pm \frac{14}{11}\sqrt{11}$.

42. $x - y = 1$; (1)
$x^3 - y^3 = 61$. (2)
Dividing (2) by (1), $x^2 + xy + y^2 = 61$. (3)
Squaring (1), $x^2 - 2xy + y^2 = 1$. (4)
Subtracting (4) from (3), $3xy = 60$;
whence, $xy = 20$.
Solving as in Ex. 15,
whence, $x = 5$, or -4; $y = 4$, or -5.

43. $x - y = 2;$ (1)

$x^5 - y^5 = 242.$ (2)

Dividing (2) by (1), $x^4 + x^3y + x^2y^2 + xy^3 + y^4 = 121.$ (3)

Raising (1) to 4th power,

$x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4 = 16.$ (4)

Subtracting (4) from (3),

$5x^3y - 5x^2y^2 + 5xy^3 = 105.$ (5)

Dividing by 5, $x^3y - x^2y^2 + xy^3 = 21.$ (6)

Squaring (1), $x^2 - 2xy + y^2 = 4.$ (7)

Multiplying (7) by xy, $x^3y - 2x^2y^2 + xy^3 = 4xy.$ (8)

Subtracting (7) from (8), $x^2y^2 + 4xy = 21.$ (9)

Completing square, etc., $xy = 3, \text{ or } -7.$ (10)

Combining (10) and (1), $x = 3,\ -1,\ \pm\sqrt{-6} + 1;$

and, $y = 1,\ -3,\ \pm\sqrt{-6} - 1.$

44. $x^2 + y^2 + 4 = 3xy;$ (1)

$x^4 + y^4 = 272.$ (2)

Squaring (1), $x^4 - 7x^2y^2 + y^4 = 16 - 24xy.$ (3)

Subtracting (3) from (2), $7x^2y^2 - 24xy = 256.$ (4)

Completing square, extracting root, etc., $xy = 8, \text{ or } -\frac{32}{7};$

whence, $y = \frac{8}{x}, \text{ or } -\frac{32}{7x}.$

Substituting in (1),

$x = \pm 4,\ \pm 2,\ \sqrt{\tfrac{1}{7}(-62 \pm 2\sqrt{705})},\ -\sqrt{\tfrac{1}{7}(-62 \pm 2\sqrt{705})};$

and, $y = \pm 2,\ \pm 4,\ \pm\sqrt{\tfrac{1}{7}(-62 \mp 2\sqrt{705})},\ \pm\sqrt{\tfrac{1}{7}(-62 \mp 2\sqrt{705})}.$

45. $x^2 + xy + 4y^2 = 6;$ (1)

$3x^2 + 8y^2 = 14.$ (2)

Let $x = vy;$

then (1) becomes, $v^2y^2 + vy^2 + 4y^2 = 6;$ (3)

then (2) becomes, $3v^2y^2 + 8y^2 = 14.$ (4)

From (3), $y^2 = \frac{6}{v^2 + v + 4}.$ (5)

From (4), $y^2 = \frac{14}{3v^2 + 8}.$ (6)

Equating (5) and (6), $\frac{6}{v^2 + v + 4} = \frac{14}{3v^2 + 8};$ (7)

whence, $v = 4, \text{ or } -\frac{1}{2}.$

Substituting in (5), $y = \pm\frac{1}{2}, \text{ or } \pm\frac{2}{5}\sqrt{10};$

and, $x = \pm 2, \text{ or } \mp\frac{1}{5}\sqrt{10}.$

46. $$3x^2 + 7xy = 82; \qquad (1)$$

$$x^2 + 5xy + 9y^2 = 279. \qquad (2)$$

Let $x = vy;$

then (1) becomes, $$3v^2y^2 + 7vy^2 = 82; \qquad (3)$$

and (2) becomes, $$v^2y^2 + 5vy^2 + 9y^2 = 279. \qquad (4)$$

Finding the values of y^2 from (3) and (4), and equating,

$$\frac{82}{3v^2 + 7v} = \frac{279}{v^2 + 5v + 9}; \qquad (5)$$

whence, $v = \frac{2}{5}$, or $-\frac{369}{151}$.

Substituting in (3), $y = \pm 5$, or $\pm 10\frac{1}{15}$;

and, $x = \pm 2$, or $\mp 24\frac{3}{5}$.

47. $$x^2 + y^2 = 3 + xy; \qquad (1)$$

$$x^4 + y^4 = 21 - x^2y^2. \qquad (2)$$

Squaring (1), $$x^4 + 2x^2y^2 + y^4 = 9 + 6xy + x^2y^2. \qquad (3)$$

Subtracting (3) from (2), $$6xy = 12; \qquad (4)$$

whence, $$xy = 2 \qquad (5)$$

Substituting the value of xy in (1), $$x^2 + y^2 = 5. \qquad (6)$$

Adding $2xy$ to (1), $$x^2 + 2xy + y^2 = 9. \qquad (7)$$

Extracting root, $$x + y = \pm 3. \qquad (8)$$

Solving as in Ex. 16, $x = \pm 2$, or ± 1;

and, $y = \pm 1$, or ± 2.

48. $$x^2 + xy + 2y^2 = 44; \qquad (1)$$

$$2x^2 - xy + y^2 = 16. \qquad (2)$$

Let $x = vy;$

then (1) becomes, $$v^2y^2 + vy^2 + 2y^2 = 44; \qquad (3)$$

and (2) becomes, $$2v^2y^2 - vy^2 + y^2 = 16. \qquad (4)$$

From (3) and (4) finding values of y^2, and equating,

$$\frac{44}{v^2 + v + 2} = \frac{16}{2v^2 - v + 1}; \qquad (5)$$

whence, $v = \frac{1}{2}$, or $\frac{1}{3}$.

Substituting in (3), $y = \pm 4$, or $\pm 3\sqrt{2}$;

and, $x = \pm 2$, or $\pm \sqrt{2}$.

49. $x^2 - xy + y^2 = 7;$ (1)

$x^4 + x^2y^2 + y^4 = 133.$ (2)

Dividing (2) by (1), $x^2 + xy + y^2 = 19.$ (3)

Subtracting (1) from (3), $2xy = 12;$

whence, $xy = 6.$

Proceed as in Ex. 17,

whence, $x = \pm 3$, or ± 2;

and $y = \pm 2$, or ± 3.

50. $x^2 + y^2 + x + y = 32;$ (1)

$xy = 12.$ (2)

Multiplying (2) by 2, and adding to (1),

$(x + y)^2 + (x + y) = 56.$ (3)

Completing square, etc., $x + y = 7$, or -8. (4)

From (2), $y = \frac{12}{x}.$

Substituting in (4), $x + \frac{12}{x} = 7$, or -8;

whence, $x = 4$, or 3, or -2, or -6;

and, $y = 3$, or 4, or -6, or -2.

51. $x^2 + 4y^2 = 256 - 4xy;$ (1)

$3y^2 - x^2 = 39.$ (2)

Transposing in (1), $x^2 + 4xy + 4y^2 = 256.$ (3)

Extracting square root, $x + 2y = 16$, or -16; (4)

whence, $x = 16 - 2y$, or $-16 - 2y$. (5)

Substituting in (2), completing square, etc.,

$y = \pm 5$, or ∓ 59; $x = \pm 6$, or ± 102.

52. $x + y + \sqrt{x + y} = 6;$ (1)

$x^2 + y^2 = 10.$ (2)

Completing square in (1), $x + y + \sqrt{x + y} + \frac{1}{4} = \frac{25}{4}.$ (3)

Extracting square root, etc., $\sqrt{x + y} = 2$, or -3.

Squaring, $x + y = 4$, or 9.

Solving, as in Ex. 8, $x = 3, 1, \frac{1}{2}(9 \pm \sqrt{-61})$; $y = 1, 3, \frac{1}{2}(9 \mp \sqrt{-61})$.

53. $x^2 + y^2 - (x + y) = 78;$ (1)

$xy + (x + y) = 39.$ (2)

Multiplying (2) by 2, $2xy + 2(x + y) = 78.$ (3)

Adding (1) and (3), $(x+y)^2+(x+y)=156.$ (4)

Completing square, etc., $x+y=12, \text{ or } -13.$ (5)

Subtracting (5) from (2), $xy=27, \text{ or } 52.$

Solving, as in Ex. 16, $x=9,\ 3,\ \frac{1}{2}(-13\pm\sqrt{-39});$

$y=3,\ 9,\ \frac{1}{2}(-13\mp\sqrt{-39}).$

54. $\frac{3(x+y)}{x-y}+\frac{3(x-y)}{x+y}=10;$ (1)

$x^2+y^2=45.$ (2)

Clearing (1), uniting, etc., $y^2=\frac{1}{4}x^2,$ (3)

whence, $y=\pm\frac{1}{2}x.$ (4)

Substituting in (2), $x=6,\ -6.$

Substituting in (4), $y=\pm 3,\ \mp 3.$

Page 246.

55. $x^2-xy=a^2+b^2;$ (1)

$xy-y^2=2ab.$ (2)

Subtracting (2) from (1), $(x-y)^2=(a-b)^2;$ (3)

whence, $x-y=\pm(a-b).$ (4)

Dividing (1) by (4), $x=\frac{a^2+b^2}{a-b},\ -\frac{a^2+b^2}{a-b}.$

Dividing (2) by (4), $y=\pm\frac{2ab}{a-b},\ \pm\frac{2ab}{a-b}.$

56. $x+2y=3a+b;$ (1)

$xy+y^2=2a(a+b).$ (2)

Multiplying (1) by y, $xy+2y^2=(3a+b)y.$ (3)

Subtracting (2) from (3), $y^2-(3a+b)y=-2a^2-2ab.$ (4)

Completing square, etc., $y-\frac{3a+b}{2}=\pm\frac{a-b}{2};$

whence, $y=a+b, \text{ or } 2a;$

and, $x=a-b, \text{ or } b-a.$

57. $x^2+y^2=2(a^2+b^2);$ (1)

$xy=a^2-b^2.$ (2)

Multiplying (2) by 2, $2xy=2a^2-2b^2.$ (3)

Adding (3) and (1) and extracting square root,

$x+y=\pm 2a.$ (4)

Subtracting (3) from (1) and extracting square root,

$$x - y = \pm 2b; \qquad (5)$$

whence, from (4) and (5), $x = a \pm b$, or $-a \pm b$;

and $y = a \mp b$, or $-a \mp b$.

58. $$x + y + \sqrt{x+y} = 12; \qquad (1)$$

$$x - y + \sqrt{x-y} = 2. \qquad (2)$$

Completing square, etc., with (1), $x + y = 9$, or 16. (3)

Completing square, etc., with (2), $x - y = 1$, or 4; (4)

whence, from (3) and (4), $x = 5, \frac{17}{2}, \frac{13}{2}, 10$;

and, $y = 4, \frac{15}{2}, \frac{5}{2}, 6$.

59. $$x^2 + y^2 + 4\sqrt{x^2+y^2} = 45; \qquad (1)$$

$$x^4 + y^4 = 337. \qquad (2)$$

Completing square in (1), etc., $\sqrt{x^2+y^2} = 5$, or -9. (3)

Squaring, $x^2 + y^2 = 25$, or 81. (4)

Squaring (4), $x^4 + 2x^2y^2 + y^4 = 625$, or 6561. (5)

Subtracting (2) from (5), $2x^2y^2 = 288$, or 6224. (6)

Subtracting (6) from (2), $x^4 - 2x^2y^2 + y^4 = 49$, or -5887. (7)

Extracting square root, $x^2 - y^2 = \pm 7$, or $\pm 29\sqrt{-7}$. (8)

Adding (8) and (4), etc.,

$$x = 4,\ -4,\ 3,\ -3,\qquad \sqrt{\tfrac{1}{2}(81 \pm 29\sqrt{-7})},\ -\sqrt{\tfrac{1}{2}(81 \pm 29\sqrt{-7})};$$

$$y = \pm 3,\ \pm 3,\ \pm 4,\ \pm 4,\ \pm\sqrt{\tfrac{1}{2}(81 \mp 29\sqrt{-7})},\ \pm\sqrt{\tfrac{1}{2}(81 \mp 29\sqrt{-7})}.$$

60. $$\sqrt{x} - \sqrt{y-x} = \sqrt{a-x}; \qquad (1)$$

$$2\sqrt{y-x} = 3\sqrt{a-x}. \qquad (2)$$

Multiplying (1) by 2, and adding (2), $2\sqrt{x} = 5\sqrt{a-x}$. (3)

Squaring both members of (3), $4x = 25a - 25x$;

whence, $x = \frac{25}{29}a$.

Substituting in (2), $y = \frac{34}{29}a$.

SIMULTANEOUS QUADRATIC EQUATIONS—PROBLEMS.

1. Let x = one, and y the other;

then, $x + y = 8$;

and, $xy = 12$.

Solve as in Ex. 15, page 246.

2. Let $x =$ one, and y the other;
then, $x + y = 12$;
and, $x^2 + y^2 = 104$.
Solve as in Ex. 8, page 244.

3. Let $x =$ one, and y the other;
then, $x + y = 13$;
and, $\sqrt{x} + \sqrt{y} = 5$.
Solve as in Ex. 8, page 244.

4. Let $x =$ one, and y the other;
then, $x + y = 20$;
and, $xy = 99$.
Solve as in Ex. 16, page 244.

5. Let $x =$ one, and y the other;
then, $x + y = 100$; (1)
and, $\sqrt{x} - \sqrt{y} = 2$. (2)
Squaring (2), $x - 2\sqrt{xy} + y = 4$. (3)
Subtracting (3) from (1), $2\sqrt{xy} = 96$. (4)
Adding (4) to (1), $x + 2\sqrt{xy} + y = 196$. (5)
Extracting square root, $\sqrt{x} + \sqrt{y} = \pm 14$. (6)
Adding (6) and (2), etc., $x = 36$; $y = 64$.

6. Let $x =$ one, and y the other;
then, $x - y = 2$;
and, $x^3 - y^3 = 56$.
Solve as in Ex. 17, page 244.

7. Let $x =$ one, and y the other;
then, $xy + y^2 = 84$;
and, $x^2 - xy = 16$.
Solve as in Ex. 23, page 244.

8. Let $x =$ one, and y the other;
then, $xy = 48$; (1)
$x^3 - y^3 = 37(x - y)^3$. (2)
Dividing (2) by $x - y$, $x^2 + xy + y^2 = 37\,(x^2 - 2\,xy + y^2)$. (3)
Substituting value of xy, $x^2 + 48 + y^2 = 37\,(x^2 - 96 + y^2)$. (4)

Transposing, etc., $x^2 + y^2 = 100.$ (5)

Multiplying (1) by 2, $2xy = 96.$ (6)

Subtracting (6) from (5), $x^2 - 2xy + y^2 = 4.$ (7)

Extracting root, $x - y = 2.$ (8)

Adding (6) and (5), extracting root, $x + y = 14;$ (9)

whence, $x = 8;\ y = 6.$

9. Let x and y represent the numbers;

then, $x + y = a;$

and, $x^2 + y^2 = b.$

Solve as in Ex. 8, page 244.

Page 247.

10. Let x and y represent the numbers;

then, $x + y + xy = 34;$

and, $(x^2 + y^2) - (x + y) = 42.$

Solve as in Ex. 53, page 245.

11. Let x and y represent the digits;

therefore, $10x + y$ will represent the number;

then, $x^2 + y^2 = 10x + y + xy;$ (1)

and, $10x + y + 36 = 10y + x.$ (2)

Transposing, etc., in (2), $x - y = 4.$ (3)

From (3), $x = 4 + y.$ (4)

Substituting in (1), etc., $x = 4;$ and $y = 8;$

hence, the number is 48.

12. Let $x =$ A's daily rate, and $y =$ B's;

then, $x - y = 12;$ (1)

and, $\frac{xy}{2} + \frac{y^2}{2} = 720.$ (2)

Clearing in (2), $xy + y^2 = 1440.$ (3)

From (1), $x = 12 + y.$ (4)

Substituting in (3), $12y + y^2 + y^2 = 1440.$ (5)

Completing square, etc., $y = 24$ miles, B's rate;

and, $x = 36$ miles, A's rate.

13. Let $x =$ quantity of cotton, and y of linen;

then, $\frac{12}{y} =$ the price of linen per yard;

and, $\frac{12}{y} - \frac{1}{2} =$ the price of cotton per yard;

therefore, $x - y = 32$;

and, $x\left(\frac{12}{y} - \frac{1}{2}\right) = 12.$

Solving, $x = 48$ yards; $y = 16$ yards.

14. Let $x =$ width of addition;

area of addition $= 4x^2 + 60x$;

hence, $4x^2 + 60x = 216$;

whence, $x = 3$;

therefore, width is 18 rods; length, 24 rods.

15. Let $x =$ the number of yards bought;

then, $x - 12 =$ the number of yards sold;

$\frac{147}{x} =$ price paid per yard;

$\frac{120\frac{1}{4}}{x - 12} =$ price per yard received;

therefore, $\frac{120\frac{1}{4}}{x - 12} - \frac{147}{x} = \frac{1}{4}$;

whence, $x = 49$, the number of yards;

and, $\frac{147}{x} = \$3$, price per yard.

16. Let $x =$ number of children;

then, $\frac{4}{x} =$ share of each;

and, $\frac{4}{x} - \frac{1}{50} = \frac{4}{x + 10}.$

Clearing, transposing, etc.,

$$x^2 + 10x = 2000.$$

Completing square, etc., $x = 40$, the number of children;

whence, $\frac{4}{x} = 10$ cents, the share of each.

17. Let $x =$ number of cows;

$\frac{2400}{x} =$ price per head;

then, $$(x-10)\left(\frac{2400+10x}{x}\right)=2500.$$

Clearing, transposing, etc.,

$$x^2-20x=2400.$$

Completing square, etc., $x = 60$, number of cows;

and, $\frac{2400}{x} = 40$ dollars, price per head.

Page 248.

18. Let $x =$ length of rectangle;

$\frac{2x}{3} =$ length of square;

then, $\frac{x+6}{3} =$ other side of rectangle;

and, $$\frac{4x^2}{9}-\frac{x^2+6x}{3}=40.$$

Clearing, uniting, etc., $x^2 - 18x = 360$;

whence, $x = 30$ rods;

and, $\frac{2x}{3} = 20$ rods.

19. Let $x =$ miles he rode per hour;

$x - 4 =$ miles he walked per hour;

then, $$\frac{10}{x-4}-\frac{10}{x}=\frac{5}{4}.$$

Clearing, etc., $x^2 - 4x = 32$.

Completing square, etc., $x = 8$, miles per hour he rode.

20. Let $x =$ number of John's marbles;

$y =$ number of Frank's marbles;

then, $x^2 + xy = 525$;

and, $y^2 + xy = 700$.

Solving, $x = 15$, John's marbles;

and, $y = 20$, Frank's marbles.

21. Let $10x + y =$ the number;

then, $x^2 + y^2 - 18 = 10x + y$;

and, $x + y = 13$.

Solving, $x = 6$; and $y = 7$;

whence, $10x + y = 67$, the number.

22. Let $x =$ one, and y the other;

then, $xy = 16$; (1)

and, $x^4 + y^4 = 4112$. (2)

From (1), $y = \frac{16}{x}$. (3)

Substituting (3) in (2), $x^4 + \left(\frac{16}{x}\right)^4 = 4112$.

Solving, $x = 8$;

and, $y = 2$.

23. Let $x =$ the number of hundreds in the first sum,

and $y =$ the number of hundreds in the second;

then, $\frac{x}{100} =$ the rate under the first conditions;

and, $\frac{y}{100} =$ the rate under the second conditions;

then, $x^2 + xy = 96$; (1)

and, $xy + y^2 = 160$. (2)

Solving, $x = 6$ and $y = 10$.

Therefore, the sums loaned were $600 and $1000.

24. Let $x =$ number of men in front of first column;

$y =$ number of men in front of second column;

$x + 10 =$ number of men in depth of first column;

$y + 10 =$ number of men in depth of second column;

then, $x^2 + 10x + y^2 + 10y = 1104$;

and, $x^2 + y^2 = 724$.

Solving, $x = 20$;

$x + 10 = 30$;

and, $y = 18$;

$y + 10 = 28$;

whence, $600 =$ men in first column;

and, $504 =$ men in second column.

25. Let $x =$ the circumference of larger wheel;

and, $y =$ the circumference of smaller wheel;

then, $$\frac{360}{y} - \frac{360}{x} = 6; \qquad (1)$$

and, $$\frac{360}{y+3} - \frac{360}{x+3} = 4. \qquad (2)$$

Solving, $x = 15$ feet; $y = 12$ feet.

Page 249.

26. Let x and y represent the numbers;

then, $$x + y = xy; \qquad (1)$$

$$x^2 - y^2 = xy. \qquad (2)$$

Dividing (2) by (1), $$x - y = 1. \qquad (3)$$

From (3), $$x = 1 + y. \qquad (4)$$

Substituting in (1), $$1 + y + y = y + y^2;$$

whence, $$y = \tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{5}, \text{ or } \tfrac{1}{2}(1 \pm \sqrt{5});$$

$$x = \tfrac{3}{2} \pm \tfrac{1}{2}\sqrt{5}, \text{ or } \tfrac{1}{2}(3 \pm \sqrt{5}).$$

27. Let $x =$ A's capital and $y =$ B's;

then, A's capital was equal to $9x$ dollars invested for 1 month;

and, B's capital was equal to $6y$ dollars invested for 1 month.

$$\frac{9x}{9x + 6y} = \text{A's share of gain};$$

$$\frac{6y}{9x + 6y} = \text{B's share of gain};$$

$$228 + 252 - 416 = 64, \text{ entire gain};$$

then, $$x + \frac{9x \times 64}{9x + 6y} = 228; \qquad (1)$$

and, $$x + y = 416. \qquad (2)$$

Solving, $x = 192$ dollars, A's capital;

$y = 224$ dollars, B's capital.

28. Let $x =$ the length and y the breadth;

then, $$2x + 2y = 100; \qquad (1)$$

$$xy = 589. \qquad (2)$$

Solving, as in Ex. 16, page 244, $x = 31$ rods; $y = 19$ rods.

29. Let $x =$ the number of women and y the number of men;

then, $x + y = 20$; (1)

and, $\frac{24}{y} - \frac{24}{x} = 1$. (2)

Solving, $x = 12$ women; $y = 8$ men.

30. Let $x =$ the number of gold coins and y the number of silver coins;

then, $x + y = 9$; (1)

and, $xy + \frac{xy}{100} = 20\frac{1}{5}$. (2)

Solving, $x = 5$, or 4 gold coins; $y = 4$, or 5 silver coins.

31. Let x and y represent the numbers;

then, $x - y = 15$; (1)

and, $\frac{xy}{2} = y^3$. (2)

Solving, $x = 18$; $y = 3$.

32. C................D................E.

Let $x =$ distance from D to E, the place where A overtook B;

then, $\frac{x}{5} =$ A's rate per hour;

$\frac{60 - x}{20} =$ B's rate per hour;

$60 - 2x =$ distance from C to D;

$\frac{x}{\frac{60 - x}{20}}$ or $\frac{20x}{60 - x} =$ whole number hours traveled;

$\frac{60 - 2x}{\frac{x}{5}}$ or $\frac{300 - 10x}{x} =$ number of hours it takes A to travel from C to D.

Therefore, $\frac{20x}{60 - x} - \frac{300 - 10x}{x} = 5$.

Clearing, etc., $x = 20$;

and, $60 - 2x = 20$ miles, the distance from C to D.

PROPERTIES OF QUADRATICS.

Page 255.

3. $x^2 + 3x + 2 = 0$.

Factoring, $(x + 2)(x + 1) = 0$;

whence, $x = -2$, or -1.

4. $x^2 + 7x + 12 = 0$.

Factoring, $(x + 4)(x + 3) = 0$;

whence, $x = -4$, or -3.

5. $x^2 - 4x - 21 = 0.$
Factoring, $(x - 7)(x + 3) = 0;$
whence, $x = +7$, or -3.

6. $x^2 - 7x - 18 = 0.$
Factoring, $(x - 9)(x + 2) = 0;$
whence, $x = +9$, or -2.

7. $x^2 + 6x + 8 = 0.$
Factoring, $(x + 4)(x + 2) = 0;$
whence, $x = -4$, or -2.

8. $x^2 + 12x + 32 = 0.$
Factoring, $(x + 8)(x + 4) = 0;$
whence, $x = -8$, or -4.

9. $x^2 - 10x - 39 = 0.$
Factoring, $(x - 13)(x + 3) = 0;$
whence, $x = 13$, or -3.

10. $x^2 - 12x - 64 = 0.$
Factoring, $(x - 16)(x + 4) = 0;$
whence, $x = 16$, or -4.

11. $4x^2 - 10x + 6 = 0.$
Factoring, $(2x - 3)(2x - 2) = 0;$
whence, $x = 1$, or $1\frac{1}{2}$.

12. $9x^2 - 27x + 18 = 0.$
Factoring, $(3x - 6)(3x - 3) = 0;$
whence, $x = 2$, or 1.

13. $4x^2 + 16ax + 12a^2 = 0.$
Factoring, $(2x + 6a)(2x + 2a) = 0;$
whence, $x = -3a$, or $-a$.

14. $9x^2 + 30bx + 24b^2 = 0.$
Factoring, $(3x + 6b)(3x + 4b) = 0;$
whence, $x = -2b$, or $-\frac{4b}{3}$.

17. $x^3 - 5x^2 + 2x - 10 = 0.$

Factoring, $x^2(x-5) + 2(x-5) = 0;$

$(x^2+2)(x-5) = 0;$

whence, $x = \pm\sqrt{-2}$, or 5.

18. $2x^3 + 6x^2 - 3x - 9 = 0.$

Factoring, $2x^2(x+3) - 3(x+3) = 0;$

$(2x^2-3)(x+3) = 0;$

whence, $x = \pm\frac{1}{2}\sqrt{6}$, or -3.

19. $ax^3 + 2x^2 - 5ax - 10 = 0.$

Factoring, $x^2(ax+2) - 5(ax+2) = 0;$

$(x^2-5)(ax+2) = 0;$

whence, $x = \pm\sqrt{5}$, or $\dfrac{-2}{a}$.

20. $(x-3)(x^2+3x+2) = 0.$

Factoring, $(x-3)(x+2)(x+1) = 0;$

whence, $x = 3$, or -2, or -1.

21. $(x+5)(x^2-4x-21) = 0.$

Factoring, $(x+5)(x-7)(x+3) = 0;$

whence, $x = -5$, or 7, or -3.

22. $x^3 - 27 = 0.$

Factoring, $(x-3)(x^2+3x+9) = 0;$

whence, $x = 3$, or $-\frac{3}{2} \pm \frac{3}{2}\sqrt{-3}$.

PROPORTION.

Page 267.

19. Let x and y represent the numbers;

then, $x + y = 8;$ (1)

and, $xy : x^2 + y^2 :: 15 : 34.$ (2)

By Principles 4 and 10, $2xy : x^2 + y^2 :: 30 : 34.$ (3)

By Principle 8, $x^2 + 2xy + y^2 : x^2 - 2xy + y^2 :: 64 : 4.$ (4)

By Principle 9, $x + y : x - y :: 8 : 2.$ (5)

Substituting from (1), $8 : x - y :: 8 : 2.$ (6)

By Principles 4 and 10, $1 : x - y :: 1 : 2.$ (7)

By Principle 1, $x - y = 2.$ (8)

Adding (1) and (8), etc., $x = 5$; and $y = 3$.

20. Let x and y represent the numbers;
then, $x - y = 3$; (1)
and, $xy : x^2 + y^2 :: 10 : 29$. (2)
By Principles 4 and 10, $2xy : x^2 + y^2 :: 20 : 29$. (3)
By Principle 8, $x^2 + 2xy + y^2 : x^2 - 2xy + y^2 :: 49 : 9$. (4)
By Principle 9, $x + y : x - y :: 7 : 3$. (5)
Substituting from 1, $x + y : 3 :: 7 : 3$. (6)
By Principles 4 and 10, $x + y : 1 :: 7 : 1$. (7)
By Principle 1, $x + y = 7$. (8)
Adding (1) and (8), etc., $x = 5$; $y = 2$.

21. Let x and y represent the numbers;
then, $x + y : x - y :: 7 : 1$; (1)
and, $xy : x + y :: 24 : 7$. (2)
From (1), by Principle 8, $2x : 2y :: 8 : 6$. (3)
By Principle 1, etc., $3x = 4y$. (4)
From (2), by Principle 10, $3xy : 3x + 3y :: 24 : 7$. (5)
Substituting from (4), $4y^2 : 7y :: 24 : 7$.
By Principle 1, etc., $y = 6$; $x = 8$.

22. Let x and y represent the numbers;
then, $x + y = 12$; (1)
and, $xy : x^2 + y^2 :: 2 : 5$. (2)
By Principles 4 and 10, $2xy : x^2 + y^2 :: 4 : 5$. (3)
By Principle 8, $x^2 + 2xy + y^2 : x^2 - 2xy + y^2 :: 9 : 1$. (4)
By Principle 9, $x + y : x - y :: 3 : 1$. (5)
Substituting from (1), $12 : x - y :: 3 : 1$. (6)
By Principle (1), etc., $x - y = 4$. (7)
Adding (1) and (7), etc., $x = 8$; $y = 4$.

23. Let x and y represent the numbers;
then, $x + y = 6$; (1)
and, $x^2 + y^2 : x^2 + 2xy + y^2 :: 5 : 9$. (2)
By Principles 4, 10, $2x^2 + 2y^2 : x^2 + 2xy + y^2 :: 10 : 9$. (3)
By Principle 7, $x^2 - 2xy + y^2 : x^2 + 2xy + y^2 :: 1 : 9$. (4)
By Principle 9, $x - y : x + y :: 1 : 3$. (5)
Substituting from (1), $x - y : 6 :: 1 : 3$. (6)
By Principle 1, $x - y = 2$. (7)
Adding (1) and (7), etc., $x = 4$; $y = 2$.

24. Let x and y represent the numbers;

then, $xy = 12;$ (1)

$$x^3 - y^3 : x^3 - 3x^2y + 3xy^2 - y^3 :: 37 : 1. \quad (2)$$

Dividing first couplet by $x - y$,

$$x^2 + xy + y^2 : x^2 - 2xy + y^2 :: 37 : 1. \quad (3)$$

By Principle 7, $3xy : x^2 - 2xy + y^2 :: 36 : 1.$ (4)

Substituting from (1), $36 : x^2 - 2xy + y^2 :: 36 : 1.$ (5)

By Principle 9, $1 : x - y :: 1 : 1.$ (6)

By Principle 1, etc., $x - y = 1.$ (7)

Solving, as in Ex. 15, page 244, $x = 4;\ y = 3.$

FRACTIONAL EQUATIONS SOLVED BY PROPORTION.

Page 269.

3. By Principle 8, $\dfrac{2\sqrt{6x}}{4} = \dfrac{8\sqrt{6x} - 3}{15};$ (1)

or, $\dfrac{\sqrt{6x}}{2} = \dfrac{8\sqrt{6x} - 3}{15}.$ (2)

Clearing, $15\sqrt{6x} = 16\sqrt{6x} - 6.$ (3)

Transposing, etc., $\sqrt{6x} = 6.$ (4)

Squaring, etc., $x = 6.$ (5)

4. By Principle 8, $\dfrac{2\sqrt{ax}}{2b} = \dfrac{6\sqrt{ax} + 3b}{7b};$ (1)

or, $\dfrac{\sqrt{ax}}{b} = \dfrac{6\sqrt{ax} + 3b}{7b}.$ (2)

Clearing, $7b\sqrt{ax} = 6b\sqrt{ax} + 3b^2.$ (3)

Transposing, etc., $b\sqrt{ax} = 3b^2.$ (4)

Dividing by b, $\sqrt{ax} = 3b;$ (5)

whence, $x = \dfrac{9b^2}{a}.$ (6)

5. By Principle 8, $\dfrac{2\sqrt{4x+1}}{2\sqrt{4x}} = \dfrac{10}{8};$ (1)

or, $\dfrac{\sqrt{4x+1}}{\sqrt{4x}} = \dfrac{5}{4}.$ (2)

Clearing, $4\sqrt{4x+1} = 5\sqrt{4x}.$ (3)

Squaring, $64x + 16 = 100x;$ (4)

$36x = 16;$ (5)

$x = \frac{4}{9}.$ (6)

6. By Principle 8, $$\frac{2\sqrt{a+x}}{2\sqrt{a-x}}=\frac{b+1}{b-1}; \qquad (1)$$

or, $$\frac{\sqrt{a+x}}{\sqrt{a-x}}=\frac{b+1}{b-1}. \qquad (2)$$

Squaring, $$\frac{a+x}{a-x}=\frac{b^2+2b+1}{b^2-2b+1}. \qquad (3)$$

By Principle 8, etc., $$\frac{x}{a}=\frac{2b}{b^2+1}.$$

Multiplying by a, $$x=\frac{2ab}{b^2+1}.$$

7. By Principle 8, etc., $$\frac{\sqrt{x^2+1}}{\sqrt{x^2-1}}=\frac{3}{1}. \qquad (1)$$

Squaring, $$\frac{x^2+1}{x^2-1}=\frac{9}{1}. \qquad (2)$$

By Principle 8, $$\frac{x^2}{1}=\frac{10}{8}, \text{ or } \frac{5}{4}; \qquad (3)$$

whence, $$x=\pm\sqrt{\tfrac{5}{4}}, \text{ or } \pm\tfrac{1}{2}\sqrt{5}. \qquad (4)$$

8. By Principle 8, etc.,
$$\frac{a+x}{\sqrt{2ax+x^2}}=\frac{b+1}{b-1}. \qquad (1)$$

Squaring, $$\frac{a^2+2ax+x^2}{2ax+x^2}=\frac{b^2+2b+1}{b^2-2b+1}. \qquad (2)$$

By Principle 7, $$\frac{a^2+2ax+x^2}{a^2}=\frac{b^2+2b+1}{4b}. \qquad (3)$$

Extracting square root, $$\frac{a+x}{a}=\frac{b+1}{2\sqrt{b}}. \qquad (4)$$

Multiplying by a, $$a+x=\frac{a(b+1)}{2\sqrt{b}}; \qquad (5)$$

whence, $$x=\frac{a(b+1)}{2\sqrt{b}}-a; \qquad (6)$$

or, $$x=\frac{ab+a-2a\sqrt{b}}{2\sqrt{b}};$$

or, $$x=\frac{a(b-2\sqrt{b}+1)}{2\sqrt{b}}, \text{ or } \frac{a(\sqrt{b}-1)^2}{2\sqrt{b}}.$$

9. By principle 8, $\frac{\sqrt{a^2+x^2}}{x}=\frac{b+c}{b-c}$. (1)

Squaring, $\frac{a^2+x^2}{x^2}=\frac{b^2+2bc+c^2}{b^2-2bc+c^2}$. (2)

By principle 7, $\frac{a^2}{x^2}=\frac{4bc}{b^2-2bc+c^2}$. (3)

Extracting square root, $\frac{a}{x}=\frac{2\sqrt{bc}}{b-c}$. (4)

Clearing, etc., $x=\frac{a(b-c)}{2\sqrt{bc}}$. (5)

ARITHMETICAL PROGRESSION.

Page 272.

6. $a=8,\ n=20,\ d=-4.$

Substituting in formula, $l=a+(n-1)d,$

$l=8-76=-68.$

7. $a=a,\ n=30,\ d=a;$

whence, $l=a+(30-1)a=30a.$

8. $a=\frac{7}{8},\ n=15,\ d=-\frac{1}{16};$

whence, $l=\frac{7}{8}+(15-1)\times-\frac{1}{16}=0.$

9. $a=1,\ n=n,\ d=2;$

whence, $l=1+(n-1)2=2n-1.$

10. $a=25$ cents, $n=50,\ d=3$ cents;

whence, $l=25+(50-1)3=\$1.72.$

11. $a=16\frac{1}{12}$ feet, $n=7,\ d=32\frac{1}{6}$ feet;

whence, $l=16\frac{1}{12}+(7-1)32\frac{1}{6}=209\frac{1}{12}$ feet.

Page 273.

7. $a=(a+b),\ n=9,\ d=c;$

whence, $l=(a+b)+(9-1)c=a+b+8c.$

Substituting in formula, $S=\frac{n}{2}(a+l),$

$$S=\tfrac{9}{2}(a+b+a+b+8c)=9a+9b+36c.$$

Page 274.

8.
$$\begin{aligned} a = x,\ n = n,\ d &= 2x; \\ l &= x + (n-1)2x = 2nx - x; \\ S &= \frac{n}{2}(x + 2nx - x) = n^2x. \end{aligned}$$

9.
$$\begin{aligned} a = 2,\ n = 8,\ d &= -1; \\ l &= 2 + (8-1) \times -1 = -5; \\ S &= 4(2-5) = -12. \end{aligned}$$

10.
$$\begin{aligned} a = 15 \text{ mi.},\ n = 11,\ d &= 3; \\ l &= 15 + (11-1)3 = 45; \\ S &= 5\tfrac{1}{2}(15 + 45) = 330 \text{ mi.} \end{aligned}$$

11.
$$\begin{aligned} a = 1,\ n = 12,\ d &= 1; \\ l &= 1 + (12-1)1 = 12; \\ S &= 6(1 + 12) = 78 \text{ strokes.} \end{aligned}$$

12.
$$\begin{aligned} a = 100 \text{ dollars},\ n = 21,\ d &= 6 \text{ dollars}; \\ l &= 100 + (21-1)6 = 220 \text{ dollars}; \\ S &= 10\tfrac{1}{2}(100 + 220) = 3360 \text{ dollars.} \end{aligned}$$

13.
$$\begin{aligned} a = 1 \text{ dollar},\ n = 26,\ d &= 2; \\ l &= 1 + (26-1)2 = 51; \\ S &= \tfrac{26}{2}(1 + 51) = 676 \text{ dollars.} \end{aligned}$$

14.
$$\begin{aligned} a = 20,\ n = 12,\ d &= 3; \\ l &= 20 + (12-1)3 = 53; \\ S &= \tfrac{12}{2}(20 + 53) = 438 \text{ mi.} \end{aligned}$$

15.
$$\begin{aligned} a = 30,\ n = 45,\ d &= -\tfrac{1}{4}; \\ l &= 30 - (45-1)\tfrac{1}{4} = 19; \\ S &= \tfrac{45}{2}(30 + 19) = 1102\tfrac{1}{2} \text{ mi.} \end{aligned}$$

16. Let $x =$ number of days B traveled;

then, $x + 5 =$ number of days A traveled;

and, $12x =$ distance;

$$a = 1;\ l = x + 5;\ n = x + 5;\ s = 12x;$$

$$12x = \frac{(x+5)(x+6)}{2};$$

$$x^2 - 13x = -30;$$

$$x^2 - 13x + \frac{169}{4} = \frac{49}{4};$$

$$x - \tfrac{13}{2} = \pm \tfrac{7}{2};$$

$$x = 3, \text{ or } 10;$$

$$12x = 36, \text{ or } 120 \text{ miles.}$$

17. $a = 8,\ n = 30,\ d = 8;$

$$l = 8 + (30 - 1)8 = 240;$$

$$S = \tfrac{30}{2}(8 + 240) = 3720 \text{ feet.}$$

18. $a = 16\tfrac{1}{12},\ n = 20,\ d = 32\tfrac{1}{6};$

$$l = 16\tfrac{1}{12} + (20 - 1)32\tfrac{1}{6} = 627\tfrac{3}{12};$$

$$S = \tfrac{20}{2}(16\tfrac{1}{12} + 627\tfrac{3}{12}) = 6433\tfrac{1}{3} \text{ feet.}$$

Page 278.

2. $a = 5,\ l = 92,\ S = 1455.$

Substituting in formula, $S = \frac{n}{2}(a + l);$

$$1455 = \frac{n}{2}(5 + 92);$$

whence, $n = 30.$

3. $a = 2,\ l = 30,\ S = 160.$

Substituting in formula, $S = \frac{n}{2}(a + l);$

$$160 = \frac{n}{2}(2 + 30);$$

whence, $n = 10.$

4. $a = 16,\ S = 30,\ d = -3\tfrac{1}{2}.$

Substituting in formula, $l = a + (n - 1)d;$

$$l = 16 + (n - 1) \times -3\tfrac{1}{2} = \frac{39 - 7n}{2}.$$

Substituting in formula, $S = \frac{n}{2}(a + l);$

$$30 = \frac{n}{2}\left(16 + \frac{39 - 7n}{2}\right).$$

Clearing, etc., $120 = 32n + 39n - 7n^2;$

whence, $n = 8.$

5. Let $x - y$, x, and $x + y$ represent the numbers;
then, $3x = 15;$ (1)
and, $x^2 + xy = 35.$ (2)
From (1), $x = 5.$ (3)
Substituting in (2), $25 + 5y = 35;$ (4)
whence, $y = 2;$
and 3, 5, 7 are the numbers.

6. Let $x - y$, x, and $x + y$ represent the numbers;
then, $3x = 9;$ (1)
and, $x^3 - xy^2 = 15.$ (2)
From (1), $x = 3.$ (3)
Substituting in (2), $27 - 3y^2 = 15;$ (4)
whence, $y = 2;$
and 1, 3, and 5 are the numbers.

7. Let $x - y$, x, and $x + y$ represent the numbers;
then, $3x = 18;$ (1)
and, $3x^2 + 2y^2 = 126.$ (2)
From (1), $x = 6.$ (3)
Substituting in (2), $108 + 2y^2 = 126;$ (4)
whence, $y = 3;$
hence, 3, 6, and 9 are the numbers.

8. Let $x - y$, x, and $x + y$ represent the numbers;
then, $x^2 - y^2 = 16;$ (1)
and, $3x^2 + 2y^2 = 93.$ (2)
(1) × 3, $3x^2 - 3y^2 = 48.$ (3)
(2) − (3), $5y^2 = 45;$ (4)
whence, $y = 3.$
Substituting in (1), $x = 5;$
hence, 2, 5, and 8 are the numbers.

9. Let $x - y$, x, and $x + y$ represent the numbers;
then, $x - y = 3;$ (1)
and, $x^2 - y^2 = 21.$ (2)
(2) ÷ (1), $x + y = 7;$ (3)
(1) + (3), etc., $x = 5;\ y = 2;$
hence, 3, 5, and 7 are the numbers.

10. Let $x - 3y$, $x - y$, $x + y$, and $x + 3y$ represent the numbers;

then, $4x = 10$; (1)

and, $x^4 - 10x^2y^2 + 9y^4 = 24$. (2)

From (1), $x = \frac{5}{2}$. (3)

Substituting in (2), etc., $y = \frac{1}{2}$;

hence, 1, 2, 3, and 4 are the numbers.

11. Let $x - 3y$, $x - y$, $x + y$, and $x + 3y$ represent the numbers;

then, $x^2 - 9y^2 = 27$; (1)

and, $x^2 - y^2 = 35$. (2)

(2) − (1), $y = 1$. (3)

Substituting in (2), $x = 6$; (4)

hence, 3, 5, 7, and 9 are the numbers.

12. Let $x - 3y$, $x - y$, $x + y$, and $x + 3y$ represent the numbers; and $2y$ will be the common difference;

then, $2xy + 6y^2 = 16$; (1)

and, $x^2 - y^2 = 24$. (2)

Let $x = vy$.

(1) becomes, $2vy^2 + 6y^2 = 16$. (3)

(2) becomes, $v^2y^2 - y^2 = 24$. (4)

From (3), $y^2 = \frac{16}{2v + 6}$. (5)

From 4, $y^2 = \frac{24}{v^2 - 1}$. (6)

Equating, $\frac{16}{2v + 6} = \frac{24}{v^2 - 1}$. (7)

Dividing by 8, etc., $\frac{1}{v + 3} = \frac{3}{v^2 - 1}$. (8)

Clearing, etc., $v = 5$; (9)

whence, $x = 5$; $y = 1$;

and 2, 4, 6, and 8 are the numbers.

Page 279.

13. Let $x - 2y$, $x - y$, x, $x + y$, and $x + 2y$ represent the numbers;

then, $5x = 40$; (1)

and, $5x^2 + 10y^2 = 410$. (2)

From (1), $x = 8$.

Substituting in (2), $y = 3$.

Hence, 2, 5, 8, 11, and 14 are the numbers.

14. Let $x-2y$, $x-y$, x, $x+y$, and $x+2y$ represent the numbers;

then, $$5x = 25; \qquad (1)$$

and, $$x^5 - 5x^3y^2 + 4xy^4 = 945. \qquad (2)$$

From (1), $$x = 5. \qquad (3)$$

Substituting in (2), $$y = 2. \qquad (4)$$

Hence, 1, 3, 5, 7, and 9 are the numbers.

15. Let $x-3y$, $x\quad y$, $x+y$, and $x+3y$ represent the numbers;

then, $$x^4 - 10x^2y^2 + 9y^4 = 280; \qquad (1)$$

and, $$4x^2 + 20y^2 = 166. \qquad (2)$$

(2) ÷ 4, $$x^2 + 5y^2 = 41\tfrac{1}{2}. \qquad (3)$$

Squaring, $$x^4 + 10x^2y^2 + 25y^4 = 1722\tfrac{1}{4}. \qquad (4)$$

Let $$x = vy.$$

Substituting in (1), $$v^4y^4 - 10v^2y^4 + 9y^4 = 280. \qquad (5)$$

Substituting in (4), $$v^4y^4 + 10v^2y^4 + 25y^4 = 1722\tfrac{1}{4}. \qquad (6)$$

From (5), $$y^4 = \frac{280}{v^4 - 10v^2 + 9}. \qquad (7)$$

From (6), $$y^4 = \frac{1722\frac{1}{4}}{v^4 + 10v^2 + 25}. \qquad (8)$$

Equating (7) and (8), $$\frac{280}{v^4 - 10v^2 + 9} = \frac{1722\frac{1}{4}}{v^4 + 10v^2 + 25}. \qquad (9)$$

Clearing, etc., $$v = \tfrac{11}{3};$$

whence, $$y = 1\tfrac{1}{2};\ x = 5\tfrac{1}{2}.$$

Hence, 1, 4, 7, and 10 are the numbers.

16. Let $x-y$, x, and $x+y$ represent the digits;

then, $100x - 100y + 10x + x + y$, or $111x - 99y =$ the number;

$$\frac{111x - 99y}{3x} = 26; \qquad (1)$$

$$100x - 100y + 10x + x + y + 198 = x - y + 10x + 100x + 100y. \qquad (2)$$

Transposing, etc., $$198y = 198;$$

whence, $$y = 1.$$

Substituting in (1), $$x = 3.$$

Hence, the digits are 2, 3, 4; and the number is 234.

GEOMETRICAL PROGRESSION.

Page 280.

6. $a = 2a,\ n = 7,\ r = 2a;\ l = 2a(2a)^{7-1} = 128a^7.$

7. $a = 3,\ n = 9,\ r = 2ax;\ l = 3(2ax)^{9-1} = 768a^8x^8.$

8. $a = 1,\ n = n,\ r = 2;\ l = 1(2)^{n-1} = 2^{n-1}.$

9. $a = 3,\ n = n,\ r = 4;\ l = 3(4)^{n-1} = 3 \times 4^{n-1}.$

10. $a = 3,\ n = 8,\ r = \frac{1}{3};\ l = 3(\frac{1}{3})^{8-1} = \frac{1}{729}.$

11. $a = 1,\ n = 8,\ r = 3;\ l = 1 \times 3^{8-1} = 2187$ dollars.

12. $a = 1000,\ n = 6,\ r = 2;\ l = 1000(2)^{6-1} = 32000$ dollars.

13. $a = 1,\ n = 10,\ r = 2;\ l = 1(2)^{10-1} = 512$ dollars.

Page 281.

14. $a = 3,\ n = 9,\ r = 3;\ l = 3(3)^{9-1} = \$196.83.$

15. $a = 1,\ n = 6,\ r = 5;\ l = 1(5)^{6-1} = \$31.25.$

16. $a = 4,\ n = 4,\ r = 256;\ l = 4(256)^{4-1} = 67108864$ bushels.

17. $a = 2000,\ n = 5,\ r = 1\frac{1}{10};\ l = 2000(\frac{11}{10})^{5-1} = \$2928.20.$

Page 282.

12. $a = 2, l = 512, r = 2;\ S = \dfrac{2 \times 512 - 2}{2 - 1} = 1022.$

13. $a = 2, l = 0, r = \frac{1}{2};\ S = \dfrac{\frac{1}{2} \times 0 - 2}{\frac{1}{2} - 1} = 4.$

14. $a = 6, l = 0, r = \frac{1}{2};\ S = \dfrac{\frac{1}{2} \times 0 - 6}{\frac{1}{2} - 1} = 12.$

Page 283.

15. $a = 2, l = 0, r = \frac{1}{3};\ S = \dfrac{\frac{1}{3} \times 0 - 2}{\frac{1}{3} - 1} = 3.$

16. $a = 1, l = 0, r = \dfrac{1}{x^2};\ S = \dfrac{\dfrac{1}{x^2} \times 0 - 1}{\dfrac{1}{x^2} - 1} = \dfrac{-1}{\dfrac{1 - x^2}{x^2}} = \dfrac{x^2}{x^2 - 1}.$

17. $a = x, l = 0, r = -\dfrac{y}{x};\ S = \dfrac{-\dfrac{y}{x} \times 0 - x}{-\dfrac{y}{x} - 1} = \dfrac{-x}{\dfrac{-y - x}{x}} = \dfrac{x^2}{x + y}.$

18. $a = \$2,\ n = 8,\ r = 2;\ S = \dfrac{\$2 \times 2^8 - 2}{2 - 1} = \$510.$

19. $a = \$1,\ n = 20,\ r = 2;\ S = \dfrac{\$1 \times 2^{20} - 1}{2 - 1} = \$1048575.$

20. $a = 500,\ n = 5,\ r = 3;\ S = \dfrac{500 \times 3^5 - 500}{3 - 1} = \$60500.$

21. $a = 1,\ n = 8,\ r = 2;\ S = \dfrac{1 \times 2^8 - 1}{2 - 1} = \$255.$

22. $a = 10,\ l = 0,\ r = \frac{4}{5};\ S = \dfrac{\frac{4}{5} \times 0 - 10}{\frac{4}{5} - 1} = 50$ feet.

23. $a = 1,\ r = 2,\ n = 32;\ S = \dfrac{1 \times 2^{32} - 1}{2 - 1} = \$42949672.95.$

Page 285.

1. $s = \dfrac{rl - a}{r - 1}$; whence, $a = rl - rs + s$.

2. Substituting in Ex. 1, $a = 2$.

3. Substituting in Ex. 1, $a = -1$.

4. $l = ar^{n-1}$; whence, $r^{n-1} = \dfrac{l}{a}$, $r = \sqrt[n-1]{\dfrac{l}{a}}$.

5. Substituting in Ex. 4, $r = 4$.

6. Substituting in Ex. 4, $r = 3$. Series; $\frac{2}{3}$, 2, 6, 18.

7. $l = ar^{n-1}$; whence, $a = \dfrac{l}{r^{n-1}}$.

8. Substituting in Ex. 7, $a = 5$. Series; 5, 15, 45, 135, 405, 1215.

9. $s = \dfrac{rl - a}{r - 1}$; whence, $rs - s = rl - a$, $r(s - l) = s - a$, $r = \dfrac{s - a}{s - l}$.

10. Substituting in Ex. 9, $r = \frac{1}{2}$. Series; 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$.

11. Substituting in Ex. 9, $r = 2$.

12. $s = \dfrac{rl - a}{r - 1}$; whence, $rs - s = rl - a$, $l = \dfrac{a + rs - s}{r}$.

13. Substituting in Ex. 12, $l = 1215$.

GEOMETRICAL PROGRESSION—PROBLEMS.

Page 287.

2. $n = 8,\ S = 1785,\ r = 2.$

Substituting in formula, $S = \dfrac{ar^n - a}{r - 1},\ 1785 = \dfrac{a \times 2^8 - a}{2 - 1};$

whence, $255\,a = 1785$, and $a = 7.$

3. $n = 6,\ S = 1365,\ r = 4.$

Hence, $1365 = \dfrac{a \times 4^6 - a}{4 - 1}.$ (1)

Clearing, etc., $1365\,a = 1365$; and $a = 1.$

4. $S = 1,\ a = \frac{1}{3},\ l = 0.$

Substituting in formula, $S = \dfrac{rl - a}{r - 1},\ 1 = \dfrac{r \times 0 - \frac{1}{3}}{r - 1}.$

Clearing, etc., $r - 1 = -\frac{1}{3}$; and $r = \frac{2}{3}.$

5. $a = 32,\ l = 4000,\ n = 4.$

Substituting in formula, $l = ar^{n-1},\ 4000 = 32\,r^{4-1};$

whence, $32\,r^3 = 4000$, and $r = 5.$

6. Let x^2, xy, and y^2 represent the numbers;

then, $x^2 + xy + y^2 = 13;$ (1)

and, $x^4 + x^2y^2 + y^4 = 91.$ (2)

(2) ÷ (1), $x^2 - xy + y^2 = 7.$ (3)

(3) + (1), etc., $x^2 + y^2 = 10.$

(1) − (3), etc., $xy = 3;$

whence, $x = 1$; and $y = 3.$

Hence, 1, 3, 9 are the numbers.

7. Let x^2, xy, and y^2 represent the numbers;

then, $x^3y^3 = 8;$ (1)

and, $x^4 + x^2y^2 + y^4 = 21.$ (2)

From (1), $x^2y^2 = 4.$ (3)

(2) + (3), $x^4 + 2\,x^2y^2 + y^4 = 25.$ (4)

Extracting square root, $x^2 + y^2 = 5.$

(2) − (3) × 3, $x^4 - 2\,x^2y^2 + y^4 = 9.$

Extracting square root, $x^2 - y^2 = 3;$

whence, $x^2 = 4$; $y^2 = 1.$

Hence, 4, 2, and 1 are the numbers.

8. Let x, xy, xy^2, and xy^3 represent the numbers;

then, $x + xy^2 = 10;$ (1)

and, $xy + xy^3 = 30.$ (2)

(2) ÷ (1), $y = 3.$

Substituting in (1), $x = 1.$

Hence, the numbers are 1, 3, 9, 27.

9. Let x, xy, xy^2, and xy^3 represent the numbers;

then, $x + xy + xy^2 + xy^3 = 15;$ (1)

and, $\dfrac{xy^3}{xy + xy^2} = \dfrac{4}{3}.$ (2)

(2) becomes, $\dfrac{y^2}{1 + y} = \dfrac{4}{3}.$ (3)

Clearing, etc., $3y^2 - 4y = 4;$ (4)

whence, $y = 2.$ (5)

Substituting in (1), $x = 1.$

Hence, the numbers are 1, 2, 4, 8.

10. Let x, xy, and xy^2 represent the numbers;

then, $x + xy + xy^2 = 14;$ (1)

and, $x^2y + x^2y^3 = 40.$ (2)

(1) × xy, $x^2y + x^2y^2 + x^2y^3 = 14xy.$ (3)

(3) − (2), $x^2y^2 - 14xy = -40;$ (4)

whence, $xy = 4;$

and, $x = 2.$

Hence, 2, 4, and 8 are the numbers.

11. Let x, xy, xy^2, and xy^3 represent the numbers;

then, $x + xy = 10;$ (1)

and, $xy^2 + xy^3 = 22\frac{1}{2}.$ (2)

(2) ÷ (1), $y^2 = \frac{9}{4}.$

Hence, $y = \frac{3}{2}$; and $x = 4.$

Therefore, 4, 6, 9, and $13\frac{1}{2}$ are the numbers.

12. Let x, xy, and xy^2 represent the numbers;

then, $x + xy^2 = 20;$ (1)

and, $x^2y^2 = 36.$ (2)

From (2), $x = \dfrac{6}{y}.$

Substituting in (1), $y = 3$; and $x = 2.$

Hence, 2, 6, and 18 are the numbers.

13. Let x, xy, xy^2 represent the numbers;

then, $$x^3y^3 = 216; \qquad (1)$$

and, $$x^2 + x^2y^4 = 328. \qquad (2)$$

From (1), $$xy = 6; \qquad (3)$$

$$y = \frac{6}{x}. \qquad (5)$$

Substituting (5) in (2), $x^2 + \dfrac{1296}{x^2} = 328.$

Clearing, completing square, etc.,

$$x = 18, \text{ or } 2; \; y = \tfrac{1}{3}, \text{ or } 3;$$

hence the numbers are, 2, 6, 18.

Page 288.

15. $$P = 3000, \; r = .07, \; n = 6.$$

Substituting in formula, $$p = \frac{Pr(1+r)^n}{(1+r)^n - 1},$$

we have, $$p = \frac{\$3000 \times .07 \times (1.07)^6}{(1.07)^6 - 1} = \$629.38+.$$

REVIEW.

Page 289.

14. Since the first and third terms are perfect squares, and the second term is equal to twice the product of the square roots of the other terms, the expression is a perfect square. Hence, the factors are $(2x+y)(2x+y)$.

15. Since the expression is the difference of the same even powers of the two quantities,

$$x^4 - y^4 = (x^2 + y^2)(x^2 - y^2) = (x^2 + y^2)(x + y)(x - y).$$

16. This expression is factored by the method explained in § 107, page 78.

17. This expression is factored in the same manner as Example 16.

18. This expression is factored in the same manner as Example 15.

19. Separating the expressions into their factors:

$$x^2 - y^2 = (x + y)(x - y);$$
$$x^2 - 2xy + y^2 = (x - y)(x - y);$$
$$xy - y^2 = y(x - y).$$

Therefore, the H. C. D. is $(x - y)$.

20. $$\begin{array}{r} 2x^4 - 5x^2 - 12)6x^4 + 11x^2 + 3(3 \\ \underline{6x^4 - 15x^2 - 36} \\ 26x^2 + 39 \end{array}$$

Dividing remainder by 13, $$\begin{array}{r} 2x^2 + 3)2x^4 - 5x^2 - 12(x^2 - 4 \\ \underline{2x^4 - 5x^2 - 12} \end{array}$$

Therefore the H. C. D. is $2x^2 + 3$.

21. $4x^4 - 24x^3 + 34x^2 + 12x - 18$ divided by $4x^3 - 18x^2 + 19x - 3$ equals $x - 1$, and $-2x^3 - 3x^2 + 34x - 21$ remainder; $4x^3 - 18x^2 + 19x - 3$ divided by $-2x^3 - 3x^2 + 34x - 21$ equals -2, and $-24x^2 + 87x - 45$ remainder. Dividing last remainder by -3, we have $8x^2 - 29x + 15$. Multiplying preceding divisor by -4, we have $8x^3 + 12x^2 - 136x + 84$; $8x^3 + 12x^2 - 136x + 84$ divided by $8x^2 - 29x + 15$ equals $x + 5$, and $x^2 - 6x + 9$ remainder; $8x^2 - 29x + 15$ divided by $x^2 - 6x + 9$ equals 8, and $19x - 57$ remainder. Dividing last remainder by 19, we have $x - 3$; $x^2 - 6x + 9$ divided by $x - 3$ equals $x - 3$. Hence, the H. C. D. is $x - 3$.

22. Multiplying first quantity by 2, we have $2x^4 - 8x^3 - 32x^2 + 14x + 48$; $2x^4 - 8x^3 - 32x^2 + 14x + 48$ divided by $2x^3 - 15x^2 + 9x + 40$ equals $x + 3$, and $x^3 + 4x^2 - 53x - 72$ remainder; $2x^3 - 15x^2 + 9x + 40$ divided by $x^3 + 4x^2 - 53x - 72$ equals 2, and $-23x^2 + 115x + 184$ remainder. Dividing the last remainder by -23, we have $x^2 - 5x - 8$; $x^3 + 4x^2 - 53x - 72$ divided by $x^2 - 5x - 8$ equals $x + 9$. Hence, the H. C. D. is $x^2 - 5x - 8$.

Page 290.

25. Factoring the terms, we have $\dfrac{(x-5)(x-2)}{(2x+3)(x-2)} = \dfrac{x-5}{2x+3}$.

26. The H. C. D. of the terms of the fraction is $x^2 - 2x + 5$. Dividing the terms by this, we have $\dfrac{x-2}{x+4}$.

27. The H. C. D. of the terms of the fraction is $a + b + c$. Dividing the terms by this, we have $\dfrac{a-b-c}{a+b-c}$.

28. Dividing the terms of the second fraction by -1, we have $-\dfrac{-x}{-1+x}$, or $+\dfrac{x}{x-1}$. Therefore the L. C. D. is $x^2 - 1$; whence, the value of the expression is $\dfrac{3x^2}{x^2-1}$.

29. Dividing the terms of the third fraction by -1, we have $\frac{x^2-16}{4-x^2}$. Reducing to a common denominator and simplifying, the result is $\frac{1}{x+2}$.

30. Dividing the terms of the second fraction by -1, we have $\frac{-1}{(a-b)(a-c)}$; dividing the terms of the third fraction by -1, we have $\frac{-1}{(a-c)(c-b)}$; dividing the terms of this fraction by -1, we have $\frac{1}{(a-c)(b-c)}$. From these fractions we find the common denominator to be $(a-b)(b-c)(a-c)$, and the result is 0.

31. The L. C. D. of the given fraction is $1+x^2+x^4$, and the result is 0.

32. The product of first and third quantities is $x^2-\frac{1}{x^2}$, and the second quantity multiplied by this gives $x^4-\frac{1}{x^4}$.

34. The dividend simplified becomes x, and the divisor becomes 1.

35. The dividend simplified becomes $\frac{x^2+y^2}{x^2-y^2}$, and the divisor becomes $\frac{x^4+y^4}{x^4-y^4}$; whence, the result is $\frac{(x^2+y^2)^2}{x^4+y^4}$.

Page 291.

36. Canceling equal factors in numerator and denominator, we have $\frac{x-y-z}{x+y+z}$.

37. The numerator simplified is $\frac{2x^2-3x+1}{x}$. Dividing this by the denominator, we have $x-1$.

38. Simplifying the dividend, we have $\frac{2\sqrt{xy}-x-y}{\sqrt{xy}}$. Dividing, the quotient is $\sqrt{y}-\sqrt{x}$.

42. $(\sqrt{x+y})^6=(x+y)^3=x^3+3x^2y+3xy^2+y^3$.

43. Simplifying the radicals, we have $\sqrt[3]{x^7y}=x^2\sqrt[3]{xy}$; $\sqrt[3]{8x^4y^4}=2xy\sqrt[3]{xy}$; $\sqrt[3]{xy^7}=y^2\sqrt[3]{xy}$. Hence, the sum is $(x^2+2xy+y^2)\sqrt[3]{xy}$, or $(x+y)^2\sqrt[3]{xy}$.

44. $\sqrt{3x^2z+6xyz+3y^2z}=(x+y)\sqrt{3z}$; $\sqrt{12y^2z}=2y\sqrt{3z}$. Subtracting, we have $(x-y)\sqrt{3z}$.

45. Simplifying, $6\sqrt[3]{32}=12\sqrt[3]{4}$; $6\sqrt[3]{\frac{4}{27}}=2\sqrt[3]{4}$. Subtracting, we have $10\sqrt[3]{4}$.

48. Removing first parenthesis, we have:

$$7 - 7 - 7 + (7 + x) = 7;$$

whence, $x = 7$.

49. Clearing, $ax - m - n = bx - m + n$.

Transposing, etc., $(a - b)x = 2n$;

whence, $x = \dfrac{2n}{a - b}$.

50. Clearing,

$$(a + b)x + (a - b)(a^2 - b^2) = (a - b)x + (a + b)(a^2 - b^2).$$

Transposing, etc., $2bx = 2b(a^2 - b^2)$;

whence, $x = a^2 - b^2$.

51. Clearing, $b^2cdx - abd^2 = a^2cd - abc^2x$.

Transposing, etc., $(b^2cd + abc^2)x = abd^2 + a^2cd$;

whence, $x = \dfrac{abd^2 + a^2cd}{b^2cd + abc^2} = \dfrac{ad(bd + ac)}{bc(bd + ac)} = \dfrac{ad}{bc}$.

52. The equation may be expressed as follows:

$$\frac{4x}{9} + \frac{3}{9} + \frac{7x - 29}{5x - 12} = \frac{4x}{9} + \frac{19}{18}.$$

Transposing, etc., $\dfrac{7x - 29}{5x - 12} = \dfrac{13}{18}$;

whence, $x = 6$.

53. Expanding,

$$x^2 + x - \tfrac{15}{4} - x^2 - 2x + 15 + \tfrac{3}{4} = 0.$$

Transposing, etc., $x = 12$.

54. Clearing,

$$m(x + a)^2 + n(x + b)^2 = m(x + a)(x + b) + n(x + a)(x + b).$$

Transposing,

$$m(x + a)^2 - m(x + a)(x + b) = n(x + a)(x + b) - n(x + b)^2.$$

Factoring, $m(x + a)(x + a - x - b) = n(x + b)(x + a - x - b)$.

Combining, $m(x + a)(a - b) = n(x + b)(a - b)$.

Dividing by $(a - b)$, $m(x + a) = n(x + b)$;

whence, $x = \dfrac{bn - am}{m - n}$.

Page 292.

55. Dividing numerator of first member by denominator:

$$x - 4 = 6 - x;$$

whence, $x = 5.$

56. Expanding and uniting terms,

$$2 + 20x^2 + 10x^4 = 242.$$

Completing square, etc., $x = \pm 2, \text{ or } \pm\sqrt{-6}.$

57. Clearing, $\sqrt{16 - x^2} = 3.$

Squaring, etc., $x = \pm\sqrt{7}.$

58. Squaring, $x - 32 = 256 - 32\sqrt{x} + x.$

Transposing, etc., $x = 81.$

59. Squaring, $4x + 21 = 4x + 4\sqrt{x} + 1.$

Transposing, etc., $x = 25.$

60. The equation may be written:

$$\frac{3\sqrt{x} - 4}{\sqrt{x} + 2} = \frac{3\sqrt{x} + 15}{\sqrt{x} + 40}.$$

Dividing numerators by 3,

$$\frac{\sqrt{x} - 4}{\sqrt{x} + 6} = \frac{\sqrt{x} + 15}{\sqrt{x} + 120}.$$

Art. 329, $\dfrac{\sqrt{x} - 4}{10} = \dfrac{\sqrt{x} + 15}{105};$

whence, $x = 4.$

61. Clearing, $2 + x + \sqrt{2x + x^2} = 4.$

Transposing, etc., $\sqrt{2x + x^2} = 2 - x.$

Squaring, etc., $x = \frac{2}{3}.$

62. Adding the three equations, $2x + 2y + 2z = 24;$

then, $x + y + z = 12.$

Subtracting first equation, $z = 5$, etc.

63. Clearing,

$$6x + 4y + 3z = 264; \quad (1)$$

$$x + 4y + 2z = 132; \quad (2)$$

$$8x + 2y + z = 152. \quad (3)$$

(1) − (2), $5x + z = 132.$

2 × (3) − (2), $15x = 172;$

$$x = 11\tfrac{7}{15}, \text{ etc.}$$

64. Eliminating v from (2) and (4), the equations combining *four* quantities are:

$$4y - 4z + 3w = 13; \qquad (6)$$
$$7x - 2z + 3w = 17; \qquad (1)$$
$$-3x + 5y - 2w = 8; \qquad (3)$$
$$3z + 8w = 33. \qquad (5)$$

Eliminating x from (1) and (3), the equations containing *three* quantities are:

$$35y - 6z - 5w = 107; \qquad (7)$$
$$3z + 8w = 33; \qquad (5)$$
$$4y - 4z + 3w = 13. \qquad (6)$$

Eliminating z from (5), (6), and (7), the resulting equations are:

$$35y + 11w = 173; \qquad (8)$$
$$58y - 19w = 175. \qquad (9)$$

Eliminating w from (8) and (9), we find: $y = 4$.

By substitution, the other values are found to be:

$$x = 2;\ z = 3;\ w = 3;\ v = 1.$$

65. (1) × a, $acx + ay + a^2z = a^2 + a^2c + ac;$ (4)

(4) − (2) and factoring, $(ac - c^2)x + (a-1)y = a^2 + a^2c - 2ac.$ (5)

(1) × c, $c^2x + cy + acz = ac + ac^2 + c^2.$ (6)

(3) − (6), $(ac - c^2)x + (2 - c)y = a^2 + ac - ac^2.$ (7)

(7) − (5), $(2 - c - a + 1)y = 3ac - ac^2 - a^2c;$ (8)

or, $(3 - c - a)y = ac(3 - c - a);$ (9)

whence, $y = ac.$

Substituting in (5), $(ac - c^2)x + a^2c - ac = a^2 + a^2c - 2ac.$ (10)

Transposing, etc., $(ac - c^2)x = a^2 - ac.$ (11)

Factoring, $c(a - c)x = a(a - c);$ (12)

whence, $x = \frac{a}{c}.$

Substituting in (1), $a + ac + az = a + ac + c.$

Transposing, etc., $z = \frac{c}{a}.$

66. Adding equations, $x^2 + 2xy + y^2 = 36.$ (3)

Extracting square root, $x + y = \pm 6.$ (4)

(1) ÷ (4), etc., $x = \pm 2;\ y = \pm 4.$

67. Divide equation (1) by 3, then solve as Ex. 16, page 244.

68. Multiplying (1) by 3, $3x^2y + 3xy^2 = 540.$ (3)
(2) + (3), $x^3 + 3x^2y + 3xy^2 + y^3 = 729.$ (4)
Extracting cube root, $x + y = 9.$ (5)
(1) ÷ (5), $xy = 20.$ (6)
Solve as Ex. 16, page 244.

69. Dividing (1) by $\sqrt{x} - \sqrt{y}$, $\sqrt{x} + \sqrt{y} = 8.$ (3)
Then solve as Ex. 16, page 244.

70. Clearing in (2), $x^2y - xy^2 = 16.$ (3)
(3) × 3, $3x^2y - 3xy^2 = 48.$ (4)
(1) − (4), $x^3 - 3x^2y + 3xy^2 - y^3 = 8.$
Extracting cube root, $x - y = 2.$
Substituting in (2), $xy = 8.$
Solve as Ex. 15, page 244.

71. (1) ÷ (2), $x^2 + y^2 = 41.$ (3)
(2) + (3), $2x^2 = 50;$
whence, $x = 5, -5;\ y = \pm 4, \pm 4.$

72. Let $x = vy;$
then (1) becomes, $v^2y^2 + 2y^2 = 41;$ (3)
and (2) becomes, $v^2y^2 + 2vy^2 = 33.$ (4)
From (3), $y^2 = \dfrac{41}{v^2 + 2}.$ (5)
From (4), $y^2 = \dfrac{33}{v^2 + 2v}.$ (6)
Equating (5) and (6), $\dfrac{41}{v^2 + 2} = \dfrac{33}{v^2 + 2v};$
whence, $v = \frac{3}{4}$, or $-11;$
and $y = \pm 4, \pm \frac{1}{3}\sqrt{3};\ x = \pm 3, \mp \frac{11}{3}\sqrt{3}.$

73. Transposing in (1), $x^2 + y^2 + x + y = 18.$ (3)
(2) × 2, $2xy = 12.$ (4)
(4) + (3), $x^2 + 2xy + y^2 + x + y = 30.$ (5)
Completing square, $(x + y)^2 + (x + y) + \frac{1}{4} = 30\frac{1}{4}.$
Extracting root, $(x + y) + \frac{1}{2} = \pm \frac{11}{2};$
whence, $x + y = 5$, or $-6.$
Solve as in Ex. 16, page 244.

74. Factoring (1), $xy(1+y)=12.$ (3)

Factoring (2), $x(1+y^3)=18.$ (4)

(3) ÷ (4), $\dfrac{y}{1-y+y^2}=\frac{2}{3}.$ (5)

Clearing, etc., $y=2$, or $\frac{1}{2}$; $x=2$, or 16.

75. Transposing in (1), $x^2y^2+4xy=96.$

Completing square, $x^2y^2+4xy+4=100.$

Extracting root, etc., $xy=8$, or $-12.$

Solve as in Ex. 16, page 244.

76. Completing square in (1),

$x+y+\sqrt{x+y}+\frac{1}{4}=\frac{49}{4}.$ (3)

Extracting root, etc., $\sqrt{x+y}=3$, or $-4.$ (4)

Squaring, $x+y=9$, or 16. (5)

Solve as in Ex. 18, page 244.

77. Represent $x^{\frac{1}{2}}$ by p, and $y^{\frac{1}{3}}$ by q;

then (1) becomes, $p^3+q^2=3p^2;$ (3)

and (2) becomes, $p+q=p^2.$ (4)

From (4), $q=p^2-p.$ (5)

Therefore, $q^2=p^4-2p^3+p^2.$ (6)

Substituting in (3), $p^3+p^4-2p^3+p^2=3p^2.$ (7)

Dividing by p^2, etc., $p^2-p=2.$ (8)

Completing square, etc., $p=2$, or -1; (9)

hence, $\sqrt{x}=2$, or -1; (10)

and, $x=4$ or 1; $y=8.$

Page 293.

78. Adding (1) and (2), $x^2+2xy+y^2-24(x+y)=-140.$ (3)

Completing square, $(x+y)^2-24(x+y)+144=4.$ (4)

Extracting root, etc., $x+y=14$, or 10. (5)

Substituting in (1), $xy=48$, or 0. (6)

Solve as in Ex. 16, page 244.

79. Factoring (2), $xy(x-y)=6.$ (3)

(3) × 2, $2xy(x-y)=12.$ (4)

(1) − (4), $(x-y)(x^2-2xy+y^2)=1;$ (5)

or, $(x-y)^2 = 1;$ (6)

whence, $x - y = 1.$ (7)

(3) ÷ (7), $xy = 6.$

Solve as in Ex. 15, page 244.

80. Let $3x$ represent one, then $4x$ will represent the other.

From the conditions, $12x^2 = 12(3x + 4x);$

whence, $x = 7;$

and the numbers are 21 and 28.

81. Let $x =$ time to midnight;

then, $\frac{4x}{5} =$ time past noon.

Therefore, $x + \frac{4x}{5} = 12;$

whence, $x = 6\frac{2}{3}$, hours to midnight;

and, $\frac{4x}{5} = 5\frac{1}{3}$, hours past noon;

or 20 minutes past 5 o'clock P.M.

82. Let x represent the number;

then, $3(x + 4) = \frac{8x}{2} - 8;$

whence, $x = 20.$

83. Let $3x$ represent one, then $4x$ will represent the other;

then, $3x + 9 : 4x + 9 :: 6 : 7;$

whence, $x = 3;$

and the numbers are 9 and 12.

84. Let x and y represent the numbers;

then, $x + y = 12;$ (1)

$x^2 - y^2 = 72.$ (2)

(2) ÷ (1), $x - y = 6.$ (3)

(3) + (1), etc., $x = 9;\ y = 3.$

85. Let $x =$ first part;

then, $x - 3 =$ second part;

$x + 10 =$ third part;

$x - 9 =$ fourth part;

$\underline{x + 16} =$ fifth part;

$5x + 14 = 99;$

whence, $x = 17.$

Therefore, the parts are 17, 14, 27, 8, 33.

86. Let x and y represent the numbers;

then, $xy = 6;$ (1)

and, $(x^2 + y^2) + (x + y) = 18;$ (2)

$(2) + (1) \times 2,\ (x^2 + 2xy + y^2) + x + y = 30.$ (3)

Completing square, etc., $x + y = 5.$ (4)

Solve as in Ex. 16, page 244.

87. Let x represent the number;

then, $\frac{x + 12}{12} - 12 = 12;$

whence, $x = 276.$

88. Let $x =$ number of apples and y the number of pears;

then, $\frac{x}{25} =$ cost of apples;

and, $\frac{3y}{25} =$ cost of pears;

$x + y = 200;$ (1)

$\frac{x}{25} + \frac{3y}{25} = 20;$ (2)

whence, $x = 50;\ y = 150.$

89. Let $x =$ time descending; y, the time returning;

then, $x + y = 10;$

and, $14x = 6y;$

whence, $x = 3;\ y = 7.$

Hence, the distance is 42 miles.

90. Let $x =$ A's age and $y =$ B's age;

then, $x - 3 = \frac{y - 3}{2};$ (1)

and, $x + 9 = \frac{5}{8}(y + 9);$ (2)

whence, $x = 21;\ y = 39.$

Page 294.

91. Let $x =$ each of digits;

then, $100x + 10x + x$, or $111x =$ the number;

$111x - 12x = 297;$

whence, $x = 3;$

and the number is 333.

92. Let x represent the number of eggs for 32 cents;

then, $\frac{32}{x} =$ price per egg;

and, $72 \times \frac{32}{x}$, or $\frac{2304}{x} =$ price of 6 dozen;

therefore, $\frac{2304}{x} = x;$

whence, $x = 48.$

Since 48 eggs are bought for 32 cents, the price is 8 cents per dozen.

93. Let $\frac{x}{y}$ represent the fraction;

then, $\frac{4x}{y+3} = \frac{2x}{y};$

and, $\frac{x+2}{4y} = \frac{x}{2y};$

whence, $x = 2;\ y = 3$, and $\frac{x}{y} = \frac{2}{3}.$

94. Let x represent the side of the square;

then, $x^2 =$ the area of the square;

and, $x^2 + 2x - 24 =$ the area of rectangle.

Hence, $x^2 = x^2 + 2x - 24;$

whence, $x = 12;$

and the area is 144 square yards.

95. Let x represent the number in first stable;

then, $x + \frac{x}{2} - 8 = x + 8;$

whence, $x = 32;$

and, $32 + 8 = 40$, the whole number.

96. Let x represent the side of one and y of the other;

then, $4x + 4y = 200;$ (1)

and, $x^2y + xy^2 = 30000.$ (2)

(1) ÷ 4, $x + y = 50.$ (3)

(2) ÷ (3), $xy = 600.$

Solving as in Ex. 15, page 246, $x = 30;\ y = 20.$

Hence, one contains 900 square rods; the other 400 square rods; and the cost of one is $ 180; of the other, $ 120.

97. Let x, y, and $z =$ the digits;

then, $100x + 10y + z =$ the number.

$$x^2 + y^2 + z^2 = 83; \tag{1}$$

$$y^2 - xz = 4; \tag{2}$$

$$100x + 10y + z + 396 = 100z + 10y + x. \tag{3}$$

(3) becomes, $$x - z = -4. \tag{4}$$

(1) − (2), $$x^2 + xz + z^2 = 79. \tag{5}$$

(5) − (4)², $$3xz = 63.$$

$$xz = 21; \ z = \frac{21}{x}; \tag{6}$$

Substituting (6) in (4), $x = 3$; $z = 7$; $y = 5$.

Hence the number is 357.

98. Let x represent the number B put in;

then, $\frac{18}{x} =$ price per week, per horse, under first conditions;

and, $\frac{72}{x} + 18 =$ amount paid for pasture;

$\frac{20}{x+2} =$ price per week, per horse, under second conditions;

and, $\frac{80}{x+2} + 20 =$ amount paid for pasture.

Therefore, $\frac{72}{x} + 18 = \frac{80}{x+2} + 20;$

Clearing, etc.,

$$x^2 + 6x = 72;$$

whence, $x = 6$;

and, $\frac{72}{x} + 18 = 30$, amount paid for pasture.

99. Let x and y represent the numbers;

then, $$x + y = 40;$$

and, $$2x - 3y = 15;$$

whence, $x = 27$; and $y = 13$.

Page 295.

100. Let x represent the number in the army;

then, $$3600 + \frac{x}{2} + 600 + \frac{x}{8} + \frac{x}{5} = x;$$

whence, $$x = 24000.$$

101. Let $2x$ represent the distance from A to B; then, $3x$ will represent the distance from C to D; and, $\frac{2x}{3}$ will represent the distance from B to C;

hence, $2x + 3x + \frac{2x}{3} = 34;$

whence, $x = 6;$

and, $2x = 12;\ 3x = 18;\ \frac{2x}{3} = 4.$

102. Clearing, $3 - x^2 = 2\sqrt{2}.$

Transposing, $x^2 = 3 - 2\sqrt{2};$

or, $x^2 = 1 - 2\sqrt{2} + 2.$

Extracting square root, $x = \pm(1 - \sqrt{2}).$

103. Let x represent the number of persons;

then, $\frac{12}{x} =$ the expenses of each;

hence, $\frac{12}{x} - \frac{12}{x + 4} = \frac{1}{2}.$

Clearing, etc., $x = 8.$

104. Let x represent the number of minute-spaces which the minute-hand travels before the two hands make a straight line; then, $\frac{x}{12} =$ the number the hour-hand travels in the same time; and, since they are 30 minute-spaces apart when they form a straight line, the difference in the number of minute-spaces passed over after 11 o'clock is 25. Therefore, $x - \frac{x}{12} = 25$; whence, $x = 27\frac{3}{11}$, and the time is $27\frac{3}{11}$ minutes past 11 o'clock.

105. Let $2x$ represent the breadth;

then, $5x$ represents the length;

and, $10x^2 = 4 \times 160;$

whence, $x = 8;$

and, $2x = 16$, the breadth; $5x = 40$, the length.

106. Since the squares of the parts are to each other as 25 to 16, the parts are to each other as 5 to 4. Therefore, $5x$ and $4x$ may represent the parts; and, $9x = 18$; whence, the parts are 10 and 8.

107. Substituting in general formula given on page 288,

$$p=\frac{\$2000\times .06\times 1.06^4}{1.06^4-1}=\$577.18+.$$

108. Let $3x$ and $4x$ represent the sides of the plat;

hence, $84x+144=3x^2$;

whence, $x=29.62$;

and the sides are $118.48+$ and $88.86+$.

109. $x+y=24$; (1)

Multiplying antecedents of the proportion by 2,

$2xy:x^2+y^2::\ 6:10.$ (2)

By Principle 8, Proportion, $x^2+2xy+y^2:x^2-2xy+y^2::16:\ 4.$ (3)

Extracting square root, $x+y:x-y::\ 4:\pm 2.$ (4)

Substituting from (1), $24:x-y::\ 4:\pm 2$; (5)

whence, $x-y=\pm 12.$ (6)

From (1) and (6), $x=18$, or 6; $y=6$, or 18.

Page 296.

110. $xy=320$; (1)

$x^3-y^3:(x-y)^3::61:1.$ (2)

Dividing first couplet by $(x-y)$, $x^2+xy+y^2:x^2-2xy+y^2::61:1.$ (3)

By Principle 7, Proportion, $3xy:x^2-2xy+y^2::60:1.$ (4)

Substituting from (1), $960:x^2-2xy+y^2::60:1.$ (5)

Dividing antecedents by 60, $16:x^2-2xy+y^2::\ 1:1$; (6)

whence, $x-y=\pm 4.$ (7)

Solving as in Ex. 15, page 244, $x=\pm 20$, or ± 16; $y=\pm 16$, or ± 20.

111. Let $x-3y$, $x-y$, $x+y$, and $x+3y$ represent the numbers.

Therefore, $2x-4y=20$; (1)

$2x+4y=44$; (2)

whence, the numbers are 7, 13, 19, and 25.

112. Let x represent the side of one, and y the side of the other;

then, $x-y=3$; (1)

and, $x^3-y^3=117.$ (2)

(2) ÷ (1), $x^2+xy+y^2=39.$ (3)

Squaring (2), $x^2-2xy+y^2=9.$ (4)

(3) − (4), $3xy=30.$ (5)

Solving as in Ex. 15, page 244, $x=5$; $y=2$.

113. Let x represent the cost of the goods;

then, $\frac{x}{100} =$ the gain per cent;

and, $x + \frac{x^2}{100} = 56;$

whence, $x = 40.$

114. Represent the numbers by x^2, xy, and y^2;

then, $x^2 + xy + y^2 = 13;$ (1)

and, $xy(x^2 + y^2) = 30.$ (2)

From (1), $x^2 + y^2 = 13 - xy.$ (3)

From (2) $x^2 + y^2 = \frac{30}{xy}.$ (4)

Equating, $\frac{30}{xy} = 13 - xy.$ (5)

Clearing, etc., $xy = 10, \text{ or } 3.$* (6)

From (6), $x = \frac{3}{y}.$

Substituting in (4), $\frac{9}{y^2} + y^2 = 10.$

Clearing, etc., $y = 3;$

whence, $x^2 = 1;\ xy = 3;\ y^2 = 9.$

115. Let $x = vy.$

Substituting in (1), $v^2y^2 + vy^2 = 12.$ (3)

Substituting in (2), $vy^2 - 2y^2 = 1.$ (4)

From (3), $y^2 = \frac{12}{v^2 + v}.$ (5)

From (4), $y^2 = \frac{1}{v - 2}.$ (6)

Equating (5) and (6), $\frac{12}{v^2 + v} = \frac{1}{v - 2}.$ (7)

Clearing, etc., $v = 3, \text{ or } 8.$

Substituting in (6), etc., $y = \pm 1, \text{ or } \pm \frac{1}{6}\sqrt{6};\ x = \pm 3, \text{ or } \pm \frac{4}{3}\sqrt{6}.$

116. Let x and y represent the depths of boxes;

then, xy^2 and $x^2y =$ capacities of respective boxes.

Hence, $4xy^2 = 5x^2y;$ (1)

and, $xy^2 - x^2y = 20.$ (2)

From (1), $4y = 5x.$ (3)

Substituting value of y in (2), etc., $x = 4;\ y = 5.$

---

* The first value of xy gives imaginary results.

117. Let x, xy, and xy^2 represent the numbers;

then, $x + xy + xy^2 = 14$; (1)

and, $x^2 + x^2y^2 + x^2y^4 = 84.$ (2)

(2) ÷ (1), $x - xy + xy^2 = 6.$ (3)

(1) + (3), $2x + 2xy^2 = 20.$ (4)

(1) − (3), $2xy = 8.$ (5)

Substituting value of x in (4), etc., $2y^2 - 5y = -2$; (6)

whence, $y = 2$; and $x = 2$.

And the numbers are 2, 4, and 8.

118. Arranging the terms of the polynomials:

$$
\begin{array}{rl}
 & a^2x^4 + 2abx^3 + b^2x^2 + 2acx^2 + 2bcx + c^2 \,\underline{|\, ax^2 + bx + c} \\
 & \underline{a^2x^4} \\
2ax^2 + bx & \big|\, 2abx^3 + b^2x^2 \\
 & \underline{\big|\, 2abx^3 + b^2x^2} \\
2ax^2 + 2bx + c & \big|\, 2acx^2 + 2bcx + c^2 \\
 & \underline{\big|\, 2acx^2 + 2bcx + c^2}
\end{array}
$$

119. Let x, xy, and xy^2 represent the lengths;

then, $x + xy + xy^2 = 70$; (1)

and, $xy^2 - x = 30.$ (2)

From (1), $x = \dfrac{70}{1 + y + y^2}.$ (3)

From (2), $x = \dfrac{30}{y^2 - 1}.$ (4)

Equating (3) and (4), $\dfrac{70}{1 + y + y^2} = \dfrac{30}{y^2 - 1}.$ (5)

Clearing, etc., $y = 2$; $x = 10.$

Therefore, 10, 20, and 40 are the lengths.

120. Let x, xy, xy^2 represent the shares;

then, $x + xy + xy^2 = 2100$; (1)

and, $xy - xy^2 = 300.$ (2)

From (1), $x = \dfrac{2100}{1 + y + y^2}.$ (3)

From (2), $x = \dfrac{300}{y - y^2}.$ (4)

Equating (3) and (4), $\dfrac{2100}{1 + y + y^2} = \dfrac{300}{y - y^2}.$ (5)

Whence, clearing, etc., $y = \frac{1}{2}$, or $\frac{1}{4}$; $x = 1200$, or 1600.

Therefore, the shares are \$1200, \$600, \$300; or \$1600, \$400, \$100.

Page 297.

121. Let x represent the quantity in less, and y the quantity in greater;

then, $x - 6 : y - 6 :: 4 : 7;$ (1)

and, $x - 3 : y - 2 :: 7 : 12.$ (2)

From (1), $7x - 42 = 4y - 24.$ (3)

From (2), $12x - 36 = 7y - 14;$ (4)

whence, $x = 38;\ y = 62.$

122. Let x represent the number of bales, and y the number of casks, the cave would hold;

then, $\frac{1}{x}$ = the portion of the cave occupied by each bale;

and, $\frac{1}{y}$ = the portion of the cave occupied by each cask.

Hence, $\frac{13}{x} + \frac{33}{y} = 1;$ (1)

$\frac{5}{x} + \frac{9}{y} = \frac{1}{3}.$ (2)

Clearing and solving, $x = 24;\ y = 72.$

123. Let $2x$ represent the price per acre;

then, $3x$ will represent the number of acres.

Hence, $\frac{6x^2 + 270}{3x} : 3x :: 3 : 2.$ (1)

From (1), $\frac{12x^2 + 540}{3x} = 9x.$

Clearing, etc., $x = 6;$

whence, the price per acre is \$ 12, and 18 is the number of acres.

124. Clearing, $\sqrt{x^6} - 40 = 3\sqrt{x^3}.$

Transposing and completing square,

$$\sqrt{x^6} - 3\sqrt{x^3} + \tfrac{9}{4} = \tfrac{169}{4}.$$

Extracting square root, $\sqrt{x^3} - \frac{3}{2} = \pm \frac{13}{2};$

whence, $\sqrt{x^3} = 8, \text{ or } -5.$

Squaring, $x^3 = 64$, or 25; whence, $x = 4$, or $\sqrt[3]{25}.$

125. Let x and y represent the numbers;

then, $x + y : x - y :: 4 : 1;$ (1)

and, $x^3 + y^3 = 152.$ (2)

From (1), $x + y = 4x - 4y;$ (3)

whence, $5y = 3x.$ (4)

Substituting value of y in (2) etc.,

$$x = 5;\ y = 3.$$

126. Let x represent B's rate;

then, $x + 3 =$ the number of hours traveled;

and, $x^2 + 3x + 8x + 24 = 204;$

whence, $x = 9;$

and, A traveled 96 miles and B 108 miles.

127. Let x represent the number of pieces;

then, $\frac{225}{x} =$ the price of each piece.

Hence, $$16x - 225 = \frac{225}{x}.$$

Clearing, etc., $$x = 15.$$

128. Let $x - y$, x, and $x + y$ represent the number;

then, $3x = 15$; and, $x = 5$; (1)

and, $x - y + 1$, $x + 4$, and $x + y + 19$ represent the numbers in geometrical progression.

From the nature of a geometrical progression we have,

$$\frac{x + 4}{x - y + 1} = \frac{x + y + 19}{x + 4}. \qquad (2)$$

Clearing, etc., $$y = 3.$$

Hence the numbers are 2, 5, and 8.

129. Let $x =$ time B required to reap the field;

and, $y =$ time C required to reap the field;

then, $\frac{1}{x} =$ part B can reap per day;

and, $\frac{1}{y} =$ part C can reap per day;

$\frac{1}{9} =$ part A can reap per day;

$\frac{1}{9} + \frac{1}{x}$, or $\frac{x + 9}{9x} =$ part A and B can reap per day.

Hence, $\frac{9x}{x + 9} =$ time in which A and B can reap the field;

$\frac{90}{x} =$ B's daily wages;

then, $$\frac{90}{x}\left(\frac{9x}{x+9}\right) = \text{sum B } would \text{ have received if C had not been called in};$$

$$\frac{90}{x} \times 5, \text{ or } \frac{450}{x} = \text{sum B } did \text{ receive.}$$

Therefore, $$\frac{90}{x}\left(\frac{9x}{x+9}\right) - \frac{450}{x} = 3\tfrac{3}{4}.$$

Clearing, etc., $$x^2 - 87x = -1080;$$

whence, $$x = 15 \text{ or } 72.$$

Since the work was completed in 5 days, C working but 2 days,

$$\frac{5}{9} + \frac{5}{x} + \frac{2}{y} = 1.$$

Substituting the values of x, and solving, $y = 18$ or $5\tfrac{1}{3}$.

Since C was an "inferior workman," he could not reap the field in less time than A. Hence the second values of y and x are inadmissible.

Therefore, B could reap the field in 15 days, and C in 18 days.

Page 298.

180. Let x and y represent the numbers;

then, $$xy = x + y; \quad (1)$$

and, $$x^2 + y^2 = x + y. \quad (2)$$

(1) × 2, $$2xy = 2x + 2y. \quad (3)$$

(2) + (3), $$x^2 + 2xy + y^2 = 3x + 3y. \quad (4)$$

(4) ÷ (x + y), $$x + y = 3. \quad (5)$$

Substituting in (2), $$x^2 + y^2 = 3. \quad (6)$$

Substituting in (3), $$2xy = 6. \quad (7)$$

(6) − (7), $$x^2 - 2xy + y^2 = -3. \quad (8)$$

Extracting square root, $$x - y = \pm\sqrt{-3}. \quad (9)$$

(5) + (9), etc., $$x = \tfrac{1}{2}(3 \pm \sqrt{-3});$$
$$y = \tfrac{1}{2}(3 \mp \sqrt{-3}).$$

181. Let x and y represent the numbers;

then, $$xy = x^2 - y^2; \quad (1)$$

and, $$x^2 + y^2 = x^3 - y^3. \quad (2)$$

Completing square in (1),

$$x^2 - xy + \frac{y^2}{4} = \frac{5y^2}{4};$$

whence, $$x = \frac{y}{2} \pm \frac{y}{2}\sqrt{5}, \text{ or } y(\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{5}). \quad (3)$$

Substituting (3) in (2),

$$\frac{y^2}{4} \pm \frac{y^2}{2}\sqrt{5} + \frac{5y^2}{4} + y^2 = \frac{y^3}{8} \pm \frac{3y^3}{8}\sqrt{5} + \frac{15y^3}{8} \pm \frac{5y^3}{8}\sqrt{5} - y^3.$$

Collecting terms, $\frac{10y^2}{4} \pm \frac{y^2}{2}\sqrt{5} = y^3 \pm y^3\sqrt{5}$.

Dividing by y^2, etc., $\frac{5}{2} \pm \frac{1}{2}\sqrt{5} = y(1 \pm \sqrt{5})$;

whence, $y = \frac{\pm\frac{1}{2}\sqrt{5} + \frac{5}{2}}{1 \pm \sqrt{5}}$, or $\pm\frac{1}{2}\sqrt{5}$.

From (3), $x = \pm\frac{1}{2}\sqrt{5}(\frac{1}{2} \pm \frac{1}{2}\sqrt{5})$, or $\frac{1}{4}(5 \pm \sqrt{5})$.

132. Let x represent the time A requires; and y represent the time B requires;

then, $x - 25 = y - 36$. (1)

Let d represent the distance from York to London;

then, $\frac{d}{x}$ represents A's rate per hour;

and, $\frac{d}{y}$ B's rate per hour;

hence, $\frac{25d}{x} + \frac{36d}{y} = d$. (2)

(2) ÷ d, $\frac{25}{x} + \frac{36}{y} = 1$. (3)

Clearing, $25y + 36x = xy$. (4)

Finding value of x in (1), and substituting in (4), we have,

$$x = 55;\ y = 66.$$

133. Let x represent the number of days.

Since each increased his rate 2 miles per day, we have two arithmetical progressions, and their sum will be the entire distance.

To find the distance A traveled, we find the sum of the series in which

$$a = 3,\ d = 2,\ n = x,\ l = 3 + 2(x - 1), \text{ or } l = 2x + 1.$$

$S = \frac{x}{2}(3 + 2x + 1)$, or $2x + x^2$, A's distance. (1)

To find B's distance, $a = 4,\ d = 2,\ n = x,\ l = 4 + 2(x - 1) = 2 + 2x$;

$S = \frac{x}{2}(4 + 2 + 2x)$, or $3x + x^2$; (2)

hence, $2x + x^2 + 3x + x^2 = 102$. (3)

Completing square, etc., $x = 6$.

134.

$$\begin{array}{r|l} & x^4 - 2x^3 + x = 30 \,\underline{|\,x^2 - x} \\ & x^4 \\ \hline 2x^2 - x & -2x^3 + x \\ & -2x^3 + x^2 \\ \hline & \qquad -x^2 + x, \text{ or } -(x^2 - x); \end{array}$$

hence, (1) becomes, $(x^2 - x)^2 - (x^2 - x) = 30.$

Completing square, etc., $x^2 - x = 6, \text{ or } -5.$

Completing square, etc., $x = 3,\ -2,\ \frac{1}{2}(1 \pm \sqrt{-19}).$

135. Multiplying by x,

$$\begin{array}{r|l} & x^4 - 6x^3 + 11x^2 - 6x = 0 \,\underline{|\,x^2 - 3x} \\ & x^4 \\ \hline 2x^2 - 3x & -6x^3 + 11x^2 - 6x \\ & -6x^3 + 9x^2 \\ \hline & \qquad 2x^2 - 6x = 2(x^2 - 3x); \end{array}$$

hence, (1) becomes, $(x^2 - 3x)^2 + 2(x^2 - 3x) = 0.$ (2)

Completing square, etc., $x^2 - 3x = 0, \text{ or } -2.$ (3)

$x = 3, \text{ or } 2, \text{ or } 1.$

136. Completing square in (1),

$(x^2 + y^2) + 4\sqrt{x^2 + y^2} + 4 = 49.$ (3)

Extracting square root, etc., $\sqrt{x^2 + y^2} = 5, \text{ or } -9.$ (4)

Squaring, disregarding 2d value, $x^2 + y^2 = 25.$ (5)

Squaring (5), $x^4 + 2x^2y^2 + y^4 = 625.$ (6)

(6) − (2), $2x^2y^2 = 288;$ (7)

whence, $xy = \pm 12;$ (8)

and, $2xy = \pm 24.$ (9)

(5) + (9), $x^2 + 2xy + y^2 = 49, \text{ or } 1;$ (10)

whence, $x + y = \pm 7, \text{ or } \pm 1.$ (11)

(5) − (9) and square root, $x - y = \pm 1, \text{ or } \pm 7;$ (12)

whence, $x = 4,\ -4,\ 3,\ -3;\ y = \pm 3,\ \pm 3,\ \pm 4,\ \pm 4.$

For roots corresponding to $\sqrt{x^2 + y^2} = -9$, see sol. of Ex. 59, p. 246.

137. Factoring (2), $xy(x^2 + y^2) = 10.$ (3)

Squaring (3), $x^2y^2(x^4 + 2x^2y^2 + y^4) = 100.$ (4)

Substituting (1) in (4), $x^2y^2(17 + 2x^2y^2) = 100;$ (5)

whence, $2x^4y^4 + 17x^2y^2 = 100.$ (6)

Completing square, etc., $xy = \pm 2 \text{ or } \pm \frac{5}{2}\sqrt{-2}.$ (7)

Substituting in (3), $x^2 + y^2 = \pm 5, \text{ or } \mp 2\sqrt{-2}.$ (8)

From (7) and (8), $\left\{\begin{array}{r} x^2+y^2=\pm 5 \\ xy=\pm 2 \end{array}\right\}$, and $\left\{\begin{array}{r} x^2+y^2=\mp 2\sqrt{-2} \\ xy=\pm \frac{5}{2}\sqrt{-2} \end{array}\right\}$.

When xy is real,

$x=\pm 2,\ \pm 1,\ \pm 2\sqrt{-1},\ \pm\sqrt{-1}$; $y=\pm 1,\ \pm 2,\ \pm\sqrt{-1},\ \pm 2\sqrt{-1}$.

When xy is imaginary,

$x=\pm \frac{1}{2}\sqrt[4]{-2}\,(3\pm\sqrt{-7})$, etc.; $y=\pm \frac{1}{2}\sqrt[4]{-2}\,(3\mp\sqrt{-7})$, etc.

138. Let $5x$ represent the original rate per hour, and y the whole distance; then, $y-10x=$ distance to be traveled after detention;

and, $\frac{y-10x}{5x}=$ time required to travel that distance at the original rate;

$\frac{y-10x}{3x}=$ time required to travel that distance at the diminished rate.

Since the train arrived $7\frac{2}{3}$ hours behind time after being detained one hour, the loss of time during running of train was $6\frac{2}{3}$ hours.

Hence,
$$\frac{y-10x}{3x}-\frac{y-10x}{5x}=6\tfrac{2}{3}. \qquad (1)$$

If the detention had occurred 50 miles further on, the distance to be traveled after the detention would have been $y-10x-50$.

Hence,
$$\frac{y-10x-50}{3x}-\frac{y-10x-50}{5x}=5\tfrac{1}{3}. \qquad (2)$$

Subtracting (2) from (1),
$$\frac{50}{3x}-\frac{50}{5x}=1\tfrac{1}{3}; \qquad (3)$$

whence, $x=5$.

Substituting in (1) $y=300$ mi., the whole distance.

IMAGINARY QUANTITIES.

Page 301.

6. $1+\sqrt{-1}$ multiplied by $1-\sqrt{-1}=1-(\sqrt{-1})^2=1+1=2$.

7. $1-\sqrt{-1}$ multiplied by $1-\sqrt{-1}=1-2\sqrt{-1}+(\sqrt{-1})^2$
$$=1-2\sqrt{-1}-1=-2\sqrt{-1}.$$

11. $\sqrt{-2}\times\sqrt{-4}=-\sqrt{8}$.
$-\sqrt{8}\times\sqrt{-8}=-8\sqrt{-1}$.

12. $7+3\sqrt{-9}$ by $6-2\sqrt{-9}=$
$$\begin{array}{l} 7+\ \ 9\sqrt{-1} \\ \underline{6-\ \ 6\sqrt{-1}} \\ 42+54\sqrt{-1} \\ \underline{\quad -42\sqrt{-1}+54} \\ 42+12\sqrt{-1}+54=96+12\sqrt{-1}. \end{array}$$

13.
$$\begin{array}{r} \sqrt{x}+\sqrt{-y} \\ \underline{\sqrt{y}+\sqrt{-x}} \\ \sqrt{xy}+y\sqrt{-1}+x\sqrt{-1} \\ \underline{-\sqrt{xy}\qquad\qquad\qquad\qquad} \\ y\sqrt{-1}+x\sqrt{-1} \end{array}$$

6. $$\begin{array}{l|l} 2 & 1+\sqrt{-1} \\ \cline{2-2} \underline{1+\sqrt{-1}} & \underline{1-\sqrt{-1}} \\ 1-\sqrt{-1} & \\ \underline{1-\sqrt{-1}} & \end{array}$$

7. $$\begin{array}{l|l} -2\sqrt{-1} & 1-\sqrt{-1} \\ \cline{2-2} \underline{-\sqrt{-1}+1} & \underline{1-\sqrt{-1}} \\ -\sqrt{-1}-1 & \\ \underline{-\sqrt{-1}-1} & \end{array}$$

Page 302.

14. $\frac{2+\sqrt{-4}}{1-\sqrt{-1}}$; rationalizing the denominator, by multiplying each term of fraction by $1+\sqrt{-1}$, we have, $\frac{4\sqrt{-1}}{2}$. Dividing the numerator by the denominator, the answer is $2\sqrt{-1}$.

15. The dividend may be written

$$\begin{array}{l|l} 2+5\sqrt{-1}+12 & 2-3\sqrt{-1} \\ \cline{2-2} \underline{2-3\sqrt{-1}} & \underline{1+4\sqrt{-1}} \\ 8\sqrt{-1}+12 & \\ \underline{8\sqrt{-1}+12} & \end{array}$$

16. $$\begin{array}{l|l} \sqrt{-6}+\sqrt{5} & \sqrt{6}-\sqrt{-5} \\ \cline{2-2} \underline{\sqrt{-6}+\sqrt{5}} & \underline{\sqrt{-1}} \end{array}$$

ZERO AND INFINITY.

Page 304.

2. Let x represent the number;

then, $$\frac{5x}{6}-4+3=\frac{x}{2}+\frac{x}{3}. \quad (1)$$

Clearing, etc., $$5x-5x=6. \quad (2)$$

Factoring, $$(5-5)x=6. \quad (3)$$

Dividing by $5-5$, $$x=\tfrac{6}{0}, \text{ or } \infty. \quad (4)$$

3. Let x represent the number in first flock;

then, $400-x=$ the number in second flock;

and, $$\frac{3x}{2}+800-2x=2x-60+860-\frac{5x}{2}. \quad (1)$$

Clearing, $$3x+1600-4x=4x-120+1720-5x. \quad (2)$$

Transposing, $$3x-4x-4x+5x=1600-1600; \quad (3)$$

whence, $$0x=0; \quad (4)$$

and, $$x=\tfrac{0}{0}, \text{ or indeterminate.}$$

4. Let $x =$ the number;

$$\frac{x}{3}-\frac{x}{4}=\frac{29x}{60}-\frac{2x}{5};$$
$$20x-15x=29x-24x;$$
$$5x=5x;$$
$$5x-5x=0;$$
$$x=\frac{0}{0}.$$

5. Let $x =$ his age;

then, $3x-10=\frac{3}{4}(4x+8)$.

Clearing, etc., $12x-12x=40+24$;

whence, $0x=64$;

and, $x=\frac{64}{0}=\infty$.

6. Let $x =$ monthly salary of each;

then, $3(10x-600)=(8x-480)3\frac{3}{4}$.

Clearing, etc., $40x-40x=2400-2400$;

whence, $0x=0$;

$$x=\frac{0}{0}.$$

INDETERMINATE EQUATIONS.

Page 308.

7. Let $17x =$ one part, and $19y$ the other;

then, $17x+19y=500$; (1)

whence, $y=\dfrac{500-17x}{19}$. (2)

Expressing value of y as a mixed quantity,

$$y=26-x+\frac{6+2x}{19}. \quad (3)$$

Since the value of y is integral, the fractional part of (3) must also be integral.

Therefore, $\dfrac{6+2x}{19}=w$ (whole number);

and, $x=\dfrac{19w-6}{2}$.

The first value of w, which makes the fraction integral is 2;

and, $x=16$; $y=12$.

Therefore, 272 and 228 are the two parts.

14. Let $x =$ number of cows;

$y =$ number of sheep;

then, $17x + 5y = 500;$ (1)

$$y = \frac{500 - 17x}{5}. \quad (2)$$

Proceeding as in Ex. 7; $x = 5, 10, 15,$ etc., $y = 83, 66, 49,$ etc.

Page 309.

17. Let $4x + 3 =$ the number of apples;

also, $5y + 1 =$ the number;

then, $4x + 3 = 5y + 1;$

whence, $$x = \frac{5y - 2}{4}.$$

Solving as Ex. 7, $y = 2, 6, 10, 14,$ etc.;

$x = 2, 7, 12, 17,$ etc.

The values which bring the number of apples between two and three dozen are 6 and 7, making the number 31.

18. Let x, y, and z represent the respective quotients;

then, $3x + 2 = 4y + 3$, or $5z + 4;$

and, $3x + 2 = 4y + 3;$ (1)

whence, $$x = \frac{4y + 1}{3}; \quad (2)$$

$4y = 3w - 1;$

and, $y = 2, 5, 8, 11, 14, 17, 20, 23, 26, 29,$ etc.,

$x = 3, 7, 11, 15, 19, 23, 27, 31, 35, 39,$ etc.

The first values which will give numbers fulfilling the conditions of the example are, 14 or 19, giving the number 59; the second values are 29, or 39, giving, 119.

19. Let x, y, and z represent the respective quotients;

then, $7x + 5 = 8y + 7$, or $9z + 8;$

and, $7x + 5 = 8y + 7.$

Solving as Ex. 7, the number is found to be 215.

20. Let x and y represent the respective quotients;

then, $39x + 16 = 56y + 27;$ (1)

$$x = \frac{56y + 11}{39}. \quad (2)$$

$$x = y + \frac{17y + 11}{39}. \quad (3)$$

Placing the fractional part equal to w, and solving as Ex. 7, the number is 1147.

21. Let $x =$ number of calves;
$y =$ number of sheep;
$z =$ number of lambs;

then, $$x + y + z = 40; \qquad (1)$$
$$7x + 5y + 3z = 192. \qquad (2)$$

Eliminating z, $$2x + y = 36; \qquad (3)$$

whence, $$x = 18 - y + \frac{y}{2};\ \frac{y}{2} = w;$$

and,
$y = 2, 4, 6, 8, 10$, etc.;
$x = 17, 16, 15, 14, 13$, etc.;
$z = 21, 20, 19, 18, 17$, etc.

22. Let $x =$ number of bushels oats;
$y =$ number of bushels barley;
$z =$ number of bushels rye;

then, $$42x + 64y + 87z = 15000; \qquad (1)$$

and, $$x + y + z = 200. \qquad (2)$$

Eliminating x, $$22y + 45z = 6600; \qquad (3)$$

whence, $$y = 300 - 2z - \frac{z}{22};\ z = 22w;$$

and,
$z = 22, 44, 66, 88, 110, 132$;
$y = 255, 210, 165, 120, 75, 30$;
$x = -77, -54, -31, -8, 15, 38$.

The first numbers fulfilling the conditions are 15, 75, 110; next, 38, 30, 132.

23. Let $x =$ number of calves;
$y =$ number of sheep;
$z =$ number of lambs;

then, $$x + y + z = 154; \qquad (1)$$
$$\frac{7x}{2} + \frac{4y}{3} + \frac{z}{2} = 154. \qquad (2)$$

Clearing (2), and uniting with (1),
$$18x + 5y = 462. \qquad (3)$$

From (3), $$y = 92 - 4x + \frac{2 + 2x}{5}.$$

Placing the fraction equal to w, and solving,
$x = 4, 9, 14$, etc.;
$y = 78, 60, 42$, etc.;
$z = 72, 85, 98$, etc.

Page 310.

24. Let $x =$ number of marbles in pile;

then, $\frac{x-1}{4} =$ A's before last division;

$\frac{3x-7}{16} =$ B's before last division;

$\frac{9x-37}{64} =$ C's before last division;

and, $\frac{27x-111}{64} =$ what was left before D threw away 1.

$$\frac{1}{4} \times \left(\frac{27x-111}{64} - 1\right) = \frac{27x-175}{256} = \text{D's}.$$

Also, add $\frac{27x-175}{256}$ to A's, B's, and C's, since they each took $\frac{1}{4}$ of the remainder.

Let any one of the four, as D's, equal y.

$$\frac{27x-175}{256} = y. \qquad (1)$$

Clearing, $27x = 175 + 256y$; (2)

whence, $x = 6 + 9y + \frac{13+13y}{27}$. (3)

Let $\frac{13+13y}{27} = w$;

whence, $y = \frac{27w-13}{13}$.

The least value of w, making the fraction a whole number is 13,

and, $y = 26$.

Substituting, $x = 253$;

and, A's $= 89$; B's $= 73$; C's $= 61$; D's $= 26$.

25. Let $x =$ number of histories;

$y =$ number of lexicons;

then, $2.50x + 1.75y = 44.75$; (1)

whence, $10x + 7y = 179$. (2)

Solve as Ex. 7, page 308.

Whence, $x = 6, 13$;

$y = 17, 7$.

26. Let $x =$ number of calves;

$y =$ number of sheep;

$z =$ number of pigs;

then, $x + y + z = 30;$ (1)

$7x + 3y + 2z = 130.$ (2)

Eliminating z from (1) and (2),

$$5x + y = 70;$$

$$x = 14 - y + \frac{4y}{5};$$

$$\frac{4y}{5} = w;$$

whence, $y = 5, 10, 15;$

$x = 13, 12, 11;$

$z = 12, 8, 4.$

27. Let $6x$, $7y$, and $8z =$ the three parts;

then, $6x + 7y + 8z = 70;$ (1)

and, $x + y + z = 10.$ (2)

Eliminating x from (1) and (2),

$$y + 2z = 10;$$

whence, $z = 5 - \frac{y}{2}.$

Solving, $y = 2, 4, 6, 8;$

$z = 4, 3, 2, 1;$

$x = 4, 3, 2, 1.$

Therefore, $6x = 24, 18, 12, 6;$

$7y = 14, 28, 42, 56;$

$8z = 32, 24, 16, 8.$

28. Let $x =$ number of sheep;

$y =$ number of pigs;

$z =$ number of lambs;

then, $x + y + z = 100;$ (1)

and, $9x + 2y + z = 400.$ (2)

Eliminating z, $8x + y = 300;$

whence, $x = 37 - y + \frac{4 + 7y}{8};$

$$\frac{4 + 7y}{8} = w.$$

Solving, $y = 4, 12, 20, 28, 36, 44,$ etc.;

$x = 37, 36, 35, 34, 33, 32,$ etc.;

$z = 59, 52, 45, 38, 31, 24,$ etc.

29. Let $x =$ number of sheep;

$y =$ number of turkeys;

$z =$ number of geese;

then, $7x + \frac{8}{3}y + z = 200;$ (1)

and, $x + y + z = 100.$ (2)

Eliminating z, $18x + 5y = 300;$ (3)

whence, $y = 60 - 4x + \frac{2x}{5};$

$$\frac{2x}{5} = w.$$

Solving, $x = 5, 10, 15;$

$y = 42, 24, 6;$

$z = 53, 66, 79.$

INEQUALITIES.

Page 313.

9. $3x + y = 9;$ (1)

$x + 2y < 12.$ (2)

$(2) \times 3 - (1)$, $5y < 27;$

whence, $y < 5\frac{2}{5}.$

Substituting in (1), $x > 1\frac{1}{5}.$

Also, $x = 2;\ y = 3.$

10. $4x + 2y = 26;$ (1)

$3x + 4y > 38.$ (2)

$(1) \times 2 - (2)$, $5x < 14;$

whence, $x < 2\frac{4}{5}.$

Substituting in (1), $y > 7\frac{2}{5}.$

Also, $x = 2;\ y = 9.$

11. $x + 6y = 13;$ (1)

$5x + 2y > 7.$ (2)

$(2) \times 3 - (1)$, $14x > 8;$

whence, $x > \frac{4}{7}.$

Substituting in (1), $y < 2\frac{1}{14}.$

Also, $x = 1;\ y = 2.$

12. Clearing, $3x + 2y = 18;$ (1)

$2x + 5y > 20.$ (2)

$(1) \times 5 - (2) \times 2$, $11x < 50;$

whence, $x < 4\frac{6}{11}.$

Substituting in (1), $y > 2\frac{2}{11}.$

Also, $x = 4;\ y = 3.$

13. Clearing,

$25x + 12y = 420;$ (1)

$15x - 8y > 80.$ (2)

$(1) \times 2 + (2) \times 3$, $95x > 1080;$

whence, $x > 11\frac{7}{19}.$

Substituting in (1), $y < 11\frac{6}{19}.$

Also, $x = 12;\ y = 10.$

14. Clearing,

$12x + 35y = 200;$ (1)

$30x - 21y < 70.$ (2)

$(1) \times 5 - (2) \times 2$, $217y > 860;$

whence, $y > 3\frac{209}{217}.$

Substituting in (1), $x < 5\frac{10}{93}.$

Also, $x = 5;\ y = 4.$

15. If a and b are unequal, $a - b > 0$. (1)

Squaring (1), $a^2 - 2ab + b^2 > 0$. (2)

Transposing, $-2ab$, $a^2 + b^2 > 2ab$.

16. Suppose a greater than b,

then, $a > b$. (1)

Dividing by b, $\frac{a}{b} > 1$. (2)

Suppose, $b > a$. (3)

Dividing by a, $\frac{b}{a} > 1$. (4)

(2) + (4), $\frac{a}{b} + \frac{b}{a} > 2$.

17. When a and b are unequal, by Example 15,

$a^2 + b^2 > 2ab$. (1)

(1) × ab, $a^3b + ab^3 > 2a^2b^2$. (2)

(2) + $(a^4 + b^4 = a^4 + b^4)$, $a^4 + a^3b + ab^3 + b^4 > a^4 + 2a^2b^2 + b^4$; (3)

$(a^3 + b^3)(a + b) > (a^2 + b^2)(a^2 + b^2)$. (4)

(4) ÷ $[(a + b)(a^2 + b^2)]$, $\frac{a^3 + b^3}{a^2 + b^2} > \frac{a^2 + b^2}{a + b}$.

LOGARITHMS.

Page 319.

9. Mantissa of log. of 483, or 4830 = .68395
Difference, 90 × .1 = 9
Logarithm of 4831 = 3.68404

10. Mantissa of log. of 384, or 3840 = .58433
Difference, 113 × .6 = 68
Logarithm of 3846 = 3.58501

11. Mantissa of log. of 278, or 2780 = .44404
Difference, 158 × .5 = 79
Logarithm of 2785 = 3.44483

12. Mantissa of log. of 316, or 3160 = .49969
Difference, 138 × .9 = 124
Logarithm of 3169 = 3.50093

13. Mantissa of log. of 187, or 1870 = .27184
Difference, 235 × .5 = 118
Logarithm of 1875 = 3.27302

14. Mantissa of log. of 234, or 2340 = .36922
Difference, 185 × .5 = 93
Logarithm of 2.345 = 0.37015

15. Mantissa of log. of 168, or 1680 = .22531
Difference, 263 × .4 = 105
Logarithm of 1.684 = 0.22636

16. Mantissa of log. of 210, or 2100 = .32222
Difference, 202 × .5 = 101
Logarithm of 2105 = 3.32323

17. Mantissa of log. of 150, or 1500 = .17609
Difference, 280 × .8 = 224
Logarithm of 1508 = 3.17833

18. Mantissa of log. of 305, or 3050 = .48430
Difference, 142 × .4 = 57
Logarithm of 3054 = 3.48487

19. Mantissa of log. of 224, or 2240 = .35025
Difference, 193 × .7 = 135
Logarithm of 2247 = 3.35160

20. Mantissa of log. of 315, or 3150 = .49831
Difference, 138 × .6 = 83
Logarithm of 3156 = 3.49914

21. Mantissa of log. of 298, or 2980 = .47422
Difference, 147 × .4 = 59
Logarithm of 2984 = 3.47481

22. Mantissa of log. of 413, or 4130 = .61595
Difference, 105 × .8 = 84
Logarithm of 4138 = 3.61679

23. Mantissa of log. of 364, or 3640 = .56110
Difference, 119 × .5 = 60
Logarithm of 3645 = 3.56170

24. Mantissa of log. of 503, or 5030 = .70157
Difference, 86 × .9 = 77
Logarithm of 5039 = 3.70234

25. Mantissa of log. of 375, or 3750 = .57403
Difference, 116 × .5 = 58
Logarithm of 3755 = 3.57461

26. Mantissa of log. of 416, or 4160 = .61909
Difference, 105 × .7 = 74
Logarithm of 4167 = 3.61983

27. Mantissa of log. of 348, or 3480 = .54158
Difference, 126 × .3 = 38
Logarithm of 3483 = 3.54196

28. Mantissa of log. of 571, or 5710 = .75664
Difference, 76 × .8 = 61
Logarithm of 5.718 = 0.75725

29. Mantissa of log. of 624, or 6240 = .79518
Difference, 69 × .5 = 35
Logarithm of 62.45 = 1.79553

30. Mantissa of log. of 896, or 8960 = .95231
Difference, 49 × .6 = 29
Logarithm of 8966 = 3.95260

31. Mantissa of log. of 784, or 7840 = .89432
Difference, 55 × .7 = 39
Logarithm of 7847 = 3.89471

32. Mantissa of log. of 352, or 3520 = .54654
Difference, 122 × .1 = 12
Logarithm of 3521 = 3.54666

33. Mantissa of log. of 471, or 4710 = .67302
Difference, 91 × .2 = 18
Logarithm of 4712 = 3.67320

34. Mantissa of log. of 836, or 8360 = .92221
Difference, 52 × .3 = 16
Logarithm of 8363 = 3.92237

35. Mantissa of log. of 927, or 9270 = .96708
Difference, 47 × .4 = 19
Logarithm of 9274 = 3.96727

36. Mantissa of log. of 499, or 4990 = .69810
Difference, 88 × .5 = 44
Logarithm of 4995 = 3.69854

37. Mantissa of log. of 672, or 6720 = .82737
Difference, 64 × .6 = 38
Logarithm of 6726 = 3.82775

38. Mantissa of log. of 578, or 5780 = .76193
Difference, 76 × .7 = 53
Logarithm of 5.787 = 0.76246

39. Mantissa of log. of 628, or 6280 = .79796
Difference, 69 × .8 = 55
Logarithm of 62.88 = 1.79851

40. Mantissa of log. of 423, or 4230 = .62634
Difference, 102 × .9 = 92
Logarithm of 423.9 = 2.62726

41. Mantissa of log. of 762, or 7620 = .88195
Logarithm of 7620 = 3.88195

42. Mantissa of log. of 834, or 8340 = .92117
Difference, 52 × .3 = 16
Logarithm of 83.43 = 1.92133

43. Mantissa of log. of 856, or 8560 = .93247
Difference, 51 × .2 = 10
Logarithm of 8.562 = 0.93257

44. Mantissa of log. of 540, or 5400 = .73239
Difference, 80 × .6 = 48
Logarithm of 540.6 = 2.73287

45. Mantissa of log. of 789, or 7890 = .89708
Difference, 55 × .9 = 50
Logarithm of 78.99 = 1.89758

46. Mantissa of log. of 875, or 8750 = .94201
Difference, 50 × .4 = 20
Logarithm of 8.754 = 0.94221

47. Mantissa of log. of 627, or 6270 = .79727
Difference, 69 × .8 = 55
Logarithm of 6278 = 3.79782

48. Mantissa of log. of 111 = .04532
Logarithm of .0111 = $\bar{2}.04532$

49. Mantissa of log. of 1 = .00000
Logarithm of .0001 = $\bar{4}.00000$

50. Mantissa of log. of 300, or 3000 = .47712
Difference, 142 × .1 = 14
Logarithm of .3001 = $\bar{1}.47726$

51. Mantissa of log. of 142 = .15229
Logarithm of .0142 = $\bar{2}.15229$

52. Mantissa of log. of 201 = .30320
Logarithm of .0201 = $\bar{2}.30320$

53. Mantissa of log. of 245, or 2450 = .38917
Difference, 177 × .1 = 18
Logarithm of 2451 = 3.38935

54. Mantissa of log. of 436, or 4360 = .63949
Difference, 100 × .7 = 70
Logarithm of 4387 = 3.64019

56. Mantissa of log. of 78 = .89209
Logarithm of .0078 = $\bar{3}.89209$

Page 320.

2. Given mantissa = .38257
Mantissa next less = .38202, number corresponding, 241.
Difference of mantissas = 55

55 ÷ tabular difference, 177 = .31; hence, the entire number is 24131. From characteristic we find there are 3 integral figures; hence, the number corresponding to logarithm is 241.31.

3. Given mantissa = .18625
Mantissa next less = .18469, number corresponding, 153.
Difference of mantissas = 156

156 ÷ 280 = .55; hence, all the figures of the number are 15355. From characteristic we know the number contains 3 integral figures; hence, 153.55 is the number.

4. Given mantissa = .23146
Mantissa next less = .23045, number corresponding, 170.
Difference of mantissas = 101

101 ÷ 248 = .4; hence, all the figures of the number are 1704. From characteristic we know the number contains 1 integral figure; hence, 1.704 is the number.

5. Given mantissa = .28643
Mantissa next less = .28556, number corresponding, 193.
Difference of mantissas = 87

87 ÷ 223 = .39; hence, all the figures of the number are 19339. Since there are no integral figures, the number is .19339.

6. Given mantissa = .98465
Mantissa next less = .98453, number corresponding, 965.
Difference of mantissas = 12

12 ÷ 45 = .26; hence, all the figures of the number are 96526. From characteristic we know that the first significant figure occupies the second decimal place; hence, .096526 is the number.

7. Given mantissa = .18425
Mantissa next less = .18184, number corresponding, 152.
Difference of mantissas = 241

241 ÷ 280 = .86; hence, all the figures of the number are 15286. Since there are 4 integral figures, the number is 1528.6.

8. Given mantissa = .86435
Mantissa next less = .86392, number corresponding, 731.
Difference of mantissas = 43

43 ÷ 59 = .72; hence, all the figures of the number are 73172. Since there are 3 integral figures, the number is 731.72.

9. Given mantissa = .24685
Mantissa next less = .24551, number corresponding, 176.
Difference of mantissas = 134

134 ÷ 248 = .54 ; hence, all the figures of the number are 17654. Since the first significant figure occupies the third decimal place, the number is .0017654.

10. Given mantissa = .98456
Mantissa next less = .98453, number corresponding, 965.
Difference of mantissas = 3

3 ÷ 45 = .06 ; hence, all the figures of the number are 96506. Since there are 3 integral figures, the number is 965.06.

11. Given mantissa = .58643
Mantissa next less = .58546, number corresponding, 385.
Difference of mantissas = 97

97 ÷ 113 = .85 ; hence, all the figures of the number are 38585. From characteristic we know that there are 3 integral figures ; hence, the number is 385.85.

12. Given mantissa = .94875
Mantissa next less = .94841, number corresponding, 888.
Difference of mantissas = 34

34 ÷ 49 = .69 ; hence, all the figures of the number are 88869. From characteristic we know that there are 4 integral figures ; hence, the number is 8886.9.

13. Given mantissa = .21654
Mantissa next less = .21484, number corresponding, 164.
Difference of mantissas = 170

170 ÷ 263 = .64 ; hence, all the figures of the number are 16464. From characteristic we know that there is 1 integral figure ; hence, the number is 1.6464.

14. Given mantissa = .72586
Mantissa next less = .72509, number corresponding, 531.
Difference of mantissas = 77

77 ÷ 81 = .95 ; hence, all the figures of the number are 53195. From characteristic we know that there are 3 integral figures ; hence, the number is 531.95.

15. Given mantissa = .21436
Mantissa next less = .21219, number corresponding, 163.
Difference of mantissas = 217

217 ÷ 263 = .82 ; hence, all the figures of the number are 16382. From characteristic there are no integral figures ; hence, the number is .16382.

16. Given mantissa = .51326
Mantissa next less = .51322, number corresponding, 326.
Difference of mantissas = 4

4 ÷ 134 = .029 ; hence, all the figures of the number are 326029. From characteristic we know that there are 3 integral figures ; hence, the number is 326.029.

17. Given mantissa = .14215
Mantissa next less = .13988, number corresponding, 138.
Difference of mantissas = 227

227 ÷ 322 = .704 ; hence, all the figures of the number are 138704. From characteristic we know that there are 4 integral figures ; hence, the number is 1387.04.

18. Given mantissa = .64823
Mantissa next less = .64738, number corresponding, 444.
Difference of mantissas = 85

85 ÷ 98 = .86; hence, all the figures of the number are 44486. From characteristic we know that the first significant figure occupies the second decimal place ; hence, the number is .044486.

19. Given mantissa = .80167
Mantissa next less = .80140, number corresponding, 633.
Difference of mantissas = 27

27 ÷ 68 = .39 ; hence, all the figures of the number are 63339. We know from characteristic that there are 5 integral figures ; hence, the number is 63339.

Page 321.

2. Logarithm of 2.3 = 0.36173
Logarithm of 3.7 = 0.56820
Number corresponding to 0.92993 is 8.51.

3. Logarithm of 25 = 1.39794
Logarithm of 3.5 = 0.54407
Number corresponding to 1.94201 is 87.5.

4. Logarithm of 216 = 2.33445
Logarithm of 3.5 = 0.54407
Number corresponding to 2.87852 is 756.

5. Logarithm of 312 = 2.49415
Logarithm of .24 = $\bar{1}.38021$
Number corresponding to 1.87436 is 74.88.

6. Logarithm of 123 = 2.08991
Logarithm of 3.4 = 0.53148
Number corresponding to 2.62139 is 418.2.

7. Logarithm of 2.24 = 0.35025
Logarithm of 2.6 = 0.41497
Number corresponding to 0.76522 is 5.824.

8. Logarithm of .0023 = $\bar{3}.36173$
Logarithm of .26 = $\bar{1}.41497$
Number corresponding to $\bar{4}.77670$ is .000598.

9. Logarithm of .0015 = $\bar{3}.17609$
Logarithm of .015 = $\bar{2}.17609$
Number corresponding to $\bar{5}.35218$ is .0000225.

10. Logarithm of .034 = $\bar{2}.53148$
Logarithm of 26 = 1.41497
Number corresponding to $\bar{1}.94645$ is .884.

11. Logarithm of .057 = $\bar{2}.75587$
Logarithm of 5.7 = 0.75587
Number corresponding to $\bar{1}.51174$ is .3249.

12. Logarithm of .068 = $\bar{2}.83251$
Logarithm of 7.2 = 0.85733
Number corresponding to $\bar{1}.68984$ is .4896.

13. Logarithm of .0018 = $\bar{3}.25527$
Logarithm of .25 = $\bar{1}.39794$
Number corresponding to $\bar{4}.65321$ is .00045.

14. Logarithm of .0432 = $\bar{2}.63548$
Logarithm of .082 = $\bar{2}.91381$
Number corresponding to $\bar{3}.54929$ is .0035424.

15. Logarithm of 856.7 = 2.93283
Logarithm of 1.38 = 0.13988
Number corresponding to 3.07271 is 1182.19.

16. Logarithm of .0796 = $\bar{2}.90091$
Logarithm of .152 = $\underline{\bar{1}.18184}$
Number corresponding to $\bar{2}.08275$ is .0120989.

17. Logarithm of 4876 = 3.68807
Logarithm of 37.5 = $\underline{1.57403}$
Number corresponding to 5.26210 is 182863+.

18. Logarithm of 3.04 = 0.48287
Logarithm of .003 = $\underline{\bar{3}.47712}$
Number corresponding to $\bar{3}.95999$ is .00912.

19. Logarithm of 468 = 2.67025
Logarithm of 34.5 = $\underline{1.53782}$
Number corresponding to 4.20807 is 16147.

20. Logarithm of .002 = $\bar{3}.30103$
Logarithm of .008 = $\underline{\bar{3}.90309}$
Number corresponding to $\bar{5}.20412$ is .000016.

21. Logarithm of 3456 = 3.53858
Logarithm of .035 = $\underline{\bar{2}.54407}$
Number corresponding to 2.08265 is 120.9.

22. Logarithm of 2874 = 3.45849
Logarithm of .836 = $\underline{\bar{1}.92221}$
Number corresponding to 3.38070 is 2402.7.

23. Logarithm of 41.03 = 1.61310
Logarithm of .507 = $\underline{\bar{1}.70501}$
Number corresponding to 1.31811 is 20.8023.

24. Logarithm of .0009 = $\bar{4}.95424$
Logarithm of .708 = $\underline{\bar{1}.85003}$
Number corresponding to $\bar{4}.80427$ is .0006372.

25. Logarithm of 8976 = 3.95308
Logarithm of 2.04 = $\underline{0.30963}$
Number corresponding to 4.26271 is 18311+.

Page 322.

2. Logarithm of 2.45 = 0.38917
Logarithm of 9.8 = 0.99123
Number corresponding to $\bar{1}.39794$ is .25.

3. Logarithm of 18.312 = 1.26273
Logarithm of 24 = 1.38021
Number corresponding to $\bar{1}.88252$ is .763.

4. Logarithm of 105.7 = 2.02409
Logarithm of 3.5 = 0.54407
Number corresponding to 1.48002 is 30.2.

5. Logarithm of 135.05 = 2.13049
Logarithm of .037 = $\bar{2}.56820$
Number corresponding to 3.56229 is 3650.

6. Logarithm of .04905 = $\bar{2}.69064$
Logarithm of .327 = $\bar{1}.51455$
Number corresponding to $\bar{1}.17609$ is .15.

7. Logarithm of 69854 = 4.84419
Logarithm of 217 = 2.33646
Number corresponding to 2.50773 is 321.9.

8. Logarithm of 823.68 = 2.91576
Logarithm of 307 = 2.48714
Number corresponding to 0.42862 is 2.683.

9. Logarithm of 34.43 = 1.53694
Logarithm of .011 = $\bar{2}.04139$
Number corresponding to 3.49555 is 3130.

10. Logarithm of 259.2 = 2.41364
Logarithm of .012 = $\bar{2}.07918$
Number corresponding to 4.33446 is 21600.

11. Logarithm of 87.36 = 1.94131
Logarithm of 2.1 = 0.32222
Number corresponding to 1.61909 is 41.6.

12. Logarithm of 97.24 = 1.98785
Logarithm of .022 = $\bar{2}.34242$
Number corresponding to 3.64543 is 4420.

13. Logarithm of 13.696 = 1.13663
Logarithm of 32 = 1.50515
Number corresponding to $\bar{1}.63148$ is .428.

14. Logarithm of 216.83 = 2.33613
Logarithm of 2.3 = 0.36173
Number corresponding to 1.97440 is 94.27.

15. Logarithm of 38142 = 4.58139
Logarithm of 3.7 = 0.56820
Number corresponding to 4.01319 is 10308.4+.

16. Logarithm of 3.485 = 0.54221
Logarithm of .205 = $\bar{1}.31175$
Number corresponding to 1.23046 is 17.

17. Logarithm of 7.865 = 0.89570
Logarithm of .013 = $\bar{2}.11394$
Number corresponding to 2.78176 is 605.

18. Logarithm of 8925 = 3.95061
Logarithm of 42.5 = 1.62839
Number corresponding to 2.32222 is 210.

19. Logarithm of 3250 = 3.51188
Logarithm of .52 = $\bar{1}.71600$
Number corresponding to 3.79588 is 6250.

2. Logarithm of 65 = 1.81291
Multiply by 2
Number corresponding to 3.62582 is 4225.

3. Logarithm of 84 = 1.92428
Multiply by 2
Number corresponding to 3.84856 is 7055.9+.

4. Logarithm of 75 = 1.87506
Multiply by 3
Number corresponding to 5.62518 is 421882+.

5. Logarithm of 64 = 1.80618
Multiply by 4
Number corresponding to 7.22472 is 16776045+.

6. Logarithm of 87 = 1.93952
Multiply by 3
Number corresponding to 5.81856 is 658500.

7. Logarithm of 37 = 1.56820
Multiply by 3
Number corresponding to 4.70460 is 50652+.

8. Logarithm of 49 = 1.69020
Multiply by 4
Number corresponding to 6.76080 is 5765000.

9. Logarithm of 91 = 1.95904
Multiply by 3
Number corresponding to 5.87712 is 753568+.

10. Logarithm of 102 = 2.00860
Multiply by 3
Number corresponding to 6.02580 is 1061183+.

11. Logarithm of 107 = 2.02938
Multiply by 4
Number corresponding to 8.11752 is 131077639+.

12. Logarithm of 146 = 2.16435
Multiply by 3
Number corresponding to 6.49305 is 3112101+.

13. Logarithm of 475 = 2.67669
Multiply by 2
Number corresponding to 5.35338 is 225621+.

14. Logarithm of 612 = 2.78675
Multiply by 2
Number corresponding to 5.57350 is 374543+.

15. Logarithm of 584 = 2.76641
Multiply by 3
Number corresponding to 8.29923 is 199170403+.

16. Logarithm of 6.75 = 0.82930
Multiply by 3
Number corresponding to 2.48790 is 307.535+.

17. Logarithm of 723 = 2.85914
Multiply by 4
Number corresponding to 11.43656 is 273253164556+.

18. Logarithm of 799 = 2.90255
Multiply by 3
Number corresponding to 8.70765 is 510095238+.

19. Logarithm of 8.76 = 0.94250
Multiply by 2
Number corresponding to 1.88500 is 76.735+.

20. Logarithm of .960 = $\bar{1}.98227$
Multiply by 3
Number corresponding to $\bar{1}.94681$ is .88473+.

21. Logarithm of 3.54 = 0.54900
Multiply by 3
Number corresponding to 1.64700 is 44.361+.

22. Logarithm of 29.3 = 1.46687
Multiply by 2
Number corresponding to 2.93374 is 858.49+.

23. Logarithm of 3.040 = 0.48287
Multiply by 3
Number corresponding to 1.44861 is 28.095+.

24. Logarithm of .0065 = $\bar{3}.81291$
Multiply by 3
Number corresponding to $\bar{7}.43873$ is .0000002746+.

25. Logarithm of .0862 = $\bar{2}.93551$
Multiply by 2
Number corresponding to $\bar{3}.87102$ is .0074305+.

26. Logarithm of 768.4 = 2.88559
Multiply by 2
Number corresponding to 5.77118 is 590452+.

27. Logarithm of 4.132 = 0.61616
Multiply by 2
Number corresponding to 1.23232 is 17.075+.

28. Logarithm of 5.184 = 0.71467
Multiply by 2
Number corresponding to 1.42934 is 26.872+.

29. Logarithm of 37.25 = 1.57112
Multiply by 3
Number corresponding to 4.71336 is 51684.52+.

Page 323.

2. Logarithm of 196 = 2.29226
Dividing by 2 = 1.14613, corresponding number, 14.

3. Logarithm of 256 = 2.40824
Dividing by 2 = 1.20412, corresponding number, 16.

4. Logarithm of 576 = 2.76042
Dividing by 2 = 1.38021, corresponding number, 24.

5. Logarithm of 1156 = 3.06297
Dividing by 2 = 1.53148, corresponding number, 34.

6. Logarithm of 1681 = 3.22557
Dividing by 2 = 1.61278, corresponding number, 41.

7. Logarithm of 3969 = 3.59869
Dividing by 2 = 1.79934, corresponding number, 63.

8. Logarithm of 4096 = 3.61236
Dividing by 3 = 1.20412, corresponding number, 16.

9. Logarithm of 5184 = 3.71467
Dividing by 2 = 1.85733, corresponding number, 72.

10. Logarithm of 13824 = 4.14065
Dividing by 3 = 1.38022, corresponding number, 24.

11. Logarithm of 15.625 = 1.19382
Dividing by 3 = .39794, corresponding number, 2.5.

12. Logarithm of 74088 = 4.86974
Dividing by 3 = 1.62325, corresponding number, 42.

13. Logarithm of 91125 = 4.95964
Dividing by 3 = 1.65321, corresponding number, 45.

14. Logarithm of 29 = 1.46240
Dividing by 3 = 0.48747, corresponding number, 3.072 +.

15. Logarithm of 65 = 1.81291
Dividing by 3 = 0.60430, corresponding number, 4.020+.

16. Logarithm of 87 = 1.93952
Dividing by 4 = 0.48488, corresponding number, 3.054+.

17. Logarithm of 96 = 1.98227
Dividing by 5 = 0.39645, corresponding number, 2.491+.

18. Logarithm of 15 = 1.17609
Dividing by 7 = 0.16801, corresponding number, 1.472+.

19. Logarithm of 67 = 1.82607
Dividing by 8 = 0.22826, corresponding number, 1.691+.

20. Logarithm of 105 = 2.02119
Dividing by 9 = 0.22458, corresponding number, 1.677+.

21. Logarithm of 111 = 2.04532
Dividing by 10 = 0.20453, corresponding number, 1.601+.

22. Logarithm of 973 = 2.98811
Dividing by 15 = 0.19921, corresponding number, 1.581+.

23. Logarithm of 3.27 = 0.51455
Dividing by 3 = 0.17152, corresponding number, 1.484+.

24. Logarithm of 9.181 = 0.96289
Dividing by 5 = 0.19258, corresponding number, 1.558+.

25. Logarithm of 62.73 = 1.79748
Dividing by 11 = 0.16341, corresponding number, 1.456+.

26. Logarithm of 3 = 0.47712
Dividing by 2 = 0.23856, corresponding number, 1.7320+.
Dividing by 3 = 0.15904, corresponding number, 1.4422+.
Logarithm of 2 = 0.30103.

Dividing by 3 = 0.10034, corresponding number, 1.2598+.
Dividing by 2 = 0.15051, corresponding number, 1.4143+.
1.7320 + 1.2598 = 2.9918;
1.4422 + 1.4143 = 2.8565;
2.9918 ÷ 2.8565 = 1.047+.

27. Logarithm of 5 = 0.69897.
Dividing by 3 = 0.23299, corresponding number, 1.7099+.
Logarithm of 8 = 0.90309.
Dividing by 4 = 0.22577, corresponding number, 1.68174+.
Logarithm of 7 = 0.84510.
Dividing by 2 = 0.42255, corresponding number, 2.6457+.
From Ex. 26, $\sqrt[3]{2} = 1.2598+$.
1.7099 − 1.6817 = .0282;
1.2598 + 2.6457 = 3.9055;
.0282 ÷ 3.9055 = .0072+.

28. From Ex. 27, $\sqrt{7} = 2.6457+$.
Logarithm of 9 = 0.95424.
Dividing by 5 = 0.19085, corresponding number, 1.5518+.
Logarithm of 12 = 1.07918.
Dividing by 3 = 0.35973, corresponding number, 2.2893+.
Logarithm of 17 = 1.23045.
Dividing by 5 = 0.24609, corresponding number, 1.7623+.
2.6457 + 1.5518 = 4.1975;
2.2893 − 1.7623 = .527;
4.1975 ÷ .527 = 7.96+.

29. Logarithm of 6 = 0.77815.
Dividing by 2 = 0.38907, corresponding number, 2.4494+.
Logarithm of 9 = 0.95424.
Dividing by 3 = 0.31808, corresponding number, 2.0800+.
Logarithm of 30 = 1.47712.
Dividing by 3 = 0.49237, corresponding number, 3.1073+.
$\sqrt[4]{49} = \sqrt{7}$ (from Ex. 27) = 2.6457+.
2.4494 + 2.0800 = 4.5294;
3.1073 − 2.6457 = .4616;
4.5294 ÷ .4616 = 9.81+.

Page 325.

2. Logarithm of A = log. 800 + 4 × log 1.06.
Logarithm of 1.06 = 0.02531
Multiply by 4
0.10124
Add logarithm of 800 2.90309
3.00433
Corresponding number = $1010, the amount.

3. Logarithm of A = log. 580 + 5 × log 1.05.
Logarithm of 1.05 = 0.02119
Multiply by 5
0.10595
Add logarithm of 580 2.76343
2.86938
Corresponding number = $740.26, the amount.

4. Logarithm of A = log. 475 + 8 × log 1.05.
Logarithm of 1.05 = 0.02119
Multiply by 8
0.16952
Add logarithm of 475 2.67669
2.84621
Corresponding number = $701.79, the amount.

5. Logarithm of P = log. 500 − 7 × log 1.06.
Logarithm of 500 = 2.69897
Logarithm of 1.06(× 7) = 0.17717
2.52180
Corresponding number = $332.51, the principal.

6. Logarithm of P = log. 800 − 8 × log 1.05.
Logarithm of 800 = 2.90309
Logarithm of 1.05(× 8) = 0.16952
2.73357
Corresponding number = $541.46, the principal.

7.
$$\begin{array}{ll}
\text{Logarithm of } P & = \log.\ 1000 - 10 \times \log 1.04. \\
\text{Logarithm of } 1000 & = 3.00000 \\
\text{Logarithm of } 1.04 (\times 10) & = \underline{0.17030} \\
 & \ 2.82970 \\
\text{Corresponding number} & = \$675.62\text{, the principal.}
\end{array}$$

8.
$$\begin{array}{ll}
\text{Logarithm of } 1 + r & = \dfrac{\log.\ 680.30 - \log.\ 500}{7}. \\
\text{Logarithm of } 680.30 & = 2.83270 \\
\text{Logarithm of } 500 & = \underline{2.69897} \\
 & \ 0.13373 \\
\text{Dividing by } 7 & = 0.01910 \\
\text{Corresponding number} & = 1.045\text{; hence the rate is } 4\tfrac{1}{2}\%.
\end{array}$$

9.
$$n = \frac{\log.\ 2000 - \log.\ 1000}{\log 1.06}.$$
$$\begin{array}{ll}
\text{Logarithm of } 2000 & = 3.30103 \\
\text{Logarithm of } 1000 & = \underline{3.00000} \\
 & \ 0.30103 \\
\text{Logarithm of } 1.06 & = 0.02531. \\
0.30103 \div 0.02531 & = 11.89 \text{ years.}
\end{array}$$

Page 327.

3.
$$A = 700 \times \frac{(1.04)^{12} - 1}{.04}.$$
$$\begin{array}{ll}
\text{By logarithms, } (1.04)^{12} & = 1.6009. \\
(1.04)^{12} - 1 & = 0.6009. \\
\text{Logarithm of } 700 & = 2.84510 \\
\text{Logarithm of } .6009 & = \underline{\bar{1}.77880} \\
 & \ 2.62390 \\
\text{Logarithm of } .04 & = \underline{\bar{2}.60206} \\
 & \ 4.02184
\end{array}$$

Corresponding number, \$10515.70, the amount.

4.
$$A = 144 \times \frac{(1.06)^{20} - 1}{.06}$$
$$\begin{array}{ll}
\text{By logarithms, } (1.06)^{20} & = 3.20783. \\
(1.06)^{20} - 1 & = 2.20783.
\end{array}$$

Logarithm of 144 $= 2.15836$
Logarithm of 2.20783 $= \underline{0.34393}$
2.50229
Logarithm of .06 $= \underline{\bar{2}.77815}$
3.72414

Corresponding number, $ 5298.19, the amount.

5.
$$P = \frac{1000}{.04} \times \frac{(1.04)^5 - 1}{(1.04)^5}.$$

By logarithms, $(1.04)^5 = 1.21678.$
$(1.04)^5 - 1 = .21678.$
Log. $\frac{1000}{.04} =$ log. 25000 $= 4.39794$
Logarithm .21678 $= \underline{\bar{1}.33603}$
3.73397
Logarithm 1.21678 $= \underline{0.08515}$
3.64882

Corresponding number, $ 4454.69, the present worth.

6.
$$A = 1200 \times \frac{(1.06)^6 - 1}{.06}.$$

By logarithms, $(1.06)^6 = 1.4188$
$(1.06)^6 - 1 = .4188$
Logarithm of 1200 $= 3.07918$
Logarithm of .4188 $= \underline{\bar{1}.62202}$
2.70120
Logarithm of .06 $= \underline{\bar{2}.77815}$
3.92305

Corresponding number, $ 8376.15, the amount.

7.
$$P = \frac{800}{.04} \times \frac{(1.04)^8 - 1}{(1.04)^8}.$$

By logarithms, $(1.04)^8 = 1.36838$
$(1.04)^8 - 1 = .36838$
Log. $\frac{800}{.04} =$ log. 20000 $= 4.30103$
Logarithm of .36838 $= \underline{\bar{1}.56630}$
3.86733
Logarithm of 1.36838 $= \underline{0.13624}$
3.73109

Corresponding number, $ 5383.82, the present worth.

3. $$P = \frac{1000}{.05} \times \frac{(1.05)^{20} - 1}{(1.05)^{20}}.$$

By logarithms, $(1.05)^{20} = 2.65335$

$(1.05)^{20} - 1 = 1.65335$

Logarithm of 1.65335 $= 0.21836$

$$\text{Log.}\frac{1000}{.05} = \text{log. } 20000 = \underline{4.30103}$$

4.51939

Logarithm, $(1.05)^{20} = \underline{0.42380}$

4.09559

Corresponding number, $12462.35, the present worth.

PERMUTATIONS AND COMBINATIONS.

Page 330.

1. $n = 9$; $r = 4$; $P_r = 9 \times 8 \times 7 \times 6 = 3024$.

2. $n = 8$; $r = 5$; $P_r = 8 \times 7 \times 6 \times 5 \times 4 = 6720$.

3. $n = 8$; $r = 4$; $C = \dfrac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2 \times 1} = 70$.

4. $n = 8$; $r = 6$; $P_r = 8 \times 7 \times 6 \times 5 \times 4 \times 3 = 20160$.

5. $n = 5$; $r = 5$; $P_r = 5 \times 4 \times 3 \times 2 \times 1 = 120$.

6. $n = 7$; $r = 7$; $P_r = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$.

Page 331.

7. $n = 12$; $r = 4$; $C = \dfrac{12 \times 11 \times 10 \times 9}{4 \times 3 \times 2 \times 1} = 495$.

8. All 5 pieces taken together give 1 sum.
Five pieces taken 2 at a time give 10 sums.
Five pieces taken 3 at a time give 10 sums.
Five pieces taken 4 at a time give 5 sums.
Five pieces taken 1 at a time give 5 sums.
Therefore, $1+10+10+5+5=31$, different sums which can be paid.

9. $n = 18$; $r = 6$; $C = \dfrac{18 \times 17 \times 16 \times 15 \times 14 \times 13}{6 \times 5 \times 4 \times 3 \times 2 \times 1} = 18564$.

10. $n = 10$; $r = 10$; $P_r = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 3628800$.

BINOMIAL THEOREM.

Page 334.

3. $(a^2 + b^3)^4 = (a^2)^4 + 4(a^2)^3(b^3) + 6(a^2)^2(b^3)^2 + 4(a^2)(b^3)^3 + (b^3)^4$
$= a^8 + 4a^6b^3 + 6a^4b^6 + 4a^2b^9 + b^{12}.$

4. $(a^{-1}+c^{-2})^4 = (a^{-1})^4 + 4(a^{-1})^3(c^{-2}) + 6(a^{-1})^2(c^{-2})^2 + 4(a^{-1})(c^{-2})^3 + (c^{-2})^4$
$= a^{-4} + 4a^{-3}c^{-2} + 6a^{-2}c^{-4} + 4a^{-1}c^{-6} + c^{-8}.$

5. $(c^{\frac{1}{2}} + d^{-2})^3 = (c^{\frac{1}{2}})^3 + 3(c^{\frac{1}{2}})^2(d^{-2}) + 3(c^{\frac{1}{2}})(d^{-2})^2 + (d^{-2})^3$
$= c^{\frac{3}{2}} + 3cd^{-2} + 3c^{\frac{1}{2}}d^{-4} + d^{-6}.$

6. $(a^{\frac{2}{3}} - x^{\frac{3}{4}})^4 = (a^{\frac{2}{3}})^4 - 4(a^{\frac{2}{3}})^3(x^{\frac{3}{4}}) + 6(a^{\frac{2}{3}})^2(x^{\frac{3}{4}})^2 - 4(a^{\frac{2}{3}})(x^{\frac{3}{4}})^3 + (x^{\frac{3}{4}})^4$
$= a^{\frac{8}{3}} - 4a^2x^{\frac{3}{4}} + 6a^{\frac{4}{3}}x^{\frac{3}{2}} - 4a^{\frac{2}{3}}x^{\frac{9}{4}} + x^3.$

7. $(x^{-\frac{1}{2}} + y^3)^5 = (x^{-\frac{1}{2}})^5 + 5(x^{-\frac{1}{2}})^4(y^3) + 10(x^{-\frac{1}{2}})^3(y^3)^2 + 10(x^{-\frac{1}{2}})^2(y^3)^3$
$+ 5(x^{-\frac{1}{2}})(y^3)^4 + (y^3)^5 = x^{-\frac{5}{2}} + 5x^{-2}y^3 + 10x^{-\frac{3}{2}}y^6 + 10x^{-1}y^9$
$+ 5x^{-\frac{1}{2}}y^{12} + y^{15}.$

8. $\left(\frac{2x}{y} - \frac{3y}{x}\right)^3 = \left(\frac{2x}{y}\right)^3 - 3\left(\frac{2x}{y}\right)^2\left(\frac{3y}{x}\right) + 3\left(\frac{2x}{y}\right)\left(\frac{3y}{x}\right)^2 - \left(\frac{3y}{x}\right)^3$
$= \frac{8x^3}{y^3} - \frac{36x}{y} + \frac{54y}{x} - \frac{27y^3}{x^3}.$

9. $(x + \sqrt{y})^4 = x^4 + 4x^3\sqrt{y} + 6x^2(\sqrt{y})^2 + 4x(\sqrt{y})^3 + (\sqrt{y})^4$
$= x^4 + 4x^3y^{\frac{1}{2}} + 6x^2y + 4xy^{\frac{3}{2}} + y^2.$

10. $(\sqrt{a^3} + 2\sqrt[3]{x})6 = (\sqrt{a^3})^6 + 6(\sqrt{a^3})^5(2\sqrt[3]{x}) + 15(\sqrt{a^3})^4(2\sqrt[3]{x})^2$
$+ 20(\sqrt{a^3})^3(2\sqrt[3]{x})^3 + 15(\sqrt{a^3})^2(2\sqrt[3]{x})^4 + 6(\sqrt{a^3})(2\sqrt[3]{x})^5 + (2\sqrt[3]{x})^6$
$= a^9 + 12a^{\frac{15}{2}}x^{\frac{1}{3}} + 60a^6x^{\frac{2}{3}} + 160a^{\frac{9}{2}}x + 240a^3x^{\frac{4}{3}} + 192a^{\frac{3}{2}}x^{\frac{5}{3}} + 64x^2.$

11. $(3x^2 - \frac{1}{3}x^{\frac{1}{2}})^5 = (3x^2)^5 - 5(3x^2)^4(\frac{1}{3}x^{\frac{1}{2}}) + 10(3x^2)^3(\frac{1}{3}x^{\frac{1}{2}})^2$
$- 10(3x^2)^2(\frac{1}{3}x^{\frac{1}{2}})^3 + 5(3x^2)(\frac{1}{3}x^{\frac{1}{2}})^4 - (\frac{1}{3}x^{\frac{1}{2}})^5 = 243x^{10} - 135x^{\frac{17}{2}}$
$+ 30x^7 - \frac{10}{3}x^{\frac{11}{2}} + \frac{5x^4}{27} - \frac{x^{\frac{5}{2}}}{243}.$

12. $$\left(\frac{2\sqrt{x}}{\sqrt[3]{y^2}}-\frac{\sqrt{y}}{\sqrt{x}}\right)^4=\left(\frac{2x^{\frac{1}{2}}}{y^{\frac{2}{3}}}-\frac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}}\right)^4=\left(\frac{2x^{\frac{1}{2}}}{y^{\frac{2}{3}}}\right)^4-4\left(\frac{2x^{\frac{1}{2}}}{y^{\frac{2}{3}}}\right)^3\left(\frac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}}\right)$$
$$+6\left(\frac{2x^{\frac{1}{2}}}{y^{\frac{2}{3}}}\right)^2\left(\frac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}}\right)^2-4\left(\frac{2x^{\frac{1}{2}}}{y^{\frac{2}{3}}}\right)\left(\frac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}}\right)^3+\left(\frac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}}\right)^4$$
$$=\frac{16x^2}{y^{\frac{8}{3}}}-\frac{32x}{y^{\frac{3}{2}}}+\frac{24}{y^{\frac{1}{3}}}-\frac{8y^{\frac{5}{6}}}{x}+\frac{y^2}{x^2}.$$

Page 335.

1. $(x+y+z)^3=[(x+y)+z]^3=(x+y)^3+3(x+y)^2z+3(x+y)z^2+z^3$
$=x^3+3x^2y+3xy^2+y^3+3x^2z+6xyz+3y^2z+3xz^2+3yz^2+z^3.$

2. $(1+x-x^2)^4=[(1+x)-x^2]^4=(1+x)^4-4(1+x)^3x^2+6(1+x)^2x^4$
$-4(1+x)x^6+x^8=1+4x+2x^2-8x^3-5x^4+8x^5+2x^6-4x^7+x^8.$

3. $(x^2-x+2)^4=[(x^2-x)+2]^4=(x^2-x)^4+4(x^2-x)^3 2+6(x^2-x)^2(2)^2$
$+4(x^2-x)(2)^3+(2)^4=x^8-4x^7+14x^6-28x^5+49x^4-56x^3+56x^2$
$-32x+16.$

4. $(a^2+ax+x^2)^3=[(a^2+ax)+x^2]^3=(a^2+ax)^3+3(a^2+ax)^2x^2$
$+3(a^2+ax)x^4+x^6=a^6+3a^5x+6a^4x^2+7a^3x^3+6a^2x^4+3ax^5+x^6.$

5. $(a+x^2+z+1)^3=[(a+x^2)+(z+1)]^3=(a+x^2)^3+3(a+x^2)^2(z+1)$
$+3(a+x^2)(z+1)^2+(z+1)^3.$

Expanding terms, and performing the multiplication indicated, the simplified answer is found.

6. $(1+2x-x^2+2)^4=(3+2x-x^2)^4=[(3+2x)-x^2]^4$
$=(3+2x)^4-4(3+2x)^3x^2+6(3+2x)^2x^4-4(3+2x)x^6+x^8.$

Expanding terms, and performing the multiplication indicated, the simplified answer is found.

UNDETERMINED COEFFICIENTS.

Page 339.

3. Assume, $$\frac{1-x}{1+x}=A+Bx+Cx^2+Dx^3+Ex^4.$$

Clearing, $$1-x=A+Bx+Cx^2+Dx^3+Ex^4$$
$$+Ax+Bx^2+Cx^3+Dx^4.$$

Equating coefficients of like powers of x,

$$A = 1;\ B + A = -1;\ C + B = 0;\ D + C = 0;\ E + D = 0;$$

whence, $A = 1;\ B = -2;\ C = 2;\ D = -2;\ E = 2.$

$$\therefore \frac{1 - x}{1 + x} = 1 - 2x + 2x^2 - 2x^3 + 2x^4.$$

4. Assume, $$\frac{1 - 2x}{1 - 3x} = A + Bx + Cx^2 + Dx^3 + Ex^4.$$

Clearing, $$\begin{aligned} 1 - 2x = A + {} & Bx + Cx^2 + Dx^3 + Ex^4 \\ & - 3Ax - 3Bx^2 - 3Cx^3 - 3Dx^4. \end{aligned}$$

Equating coefficients of like powers of x,

$$A = 1;\ B - 3A = -2;\ C - 3B = 0;\ D - 3C = 0;\ E - 3D = 0;$$

whence, $A = 1;\ B = 1;\ C = 3;\ D = 9;\ E = 27.$

$$\therefore \frac{1 - 2x}{1 - 3x} = 1 + x + 3x^2 + 9x^3 + 27x^4.$$

5. Assume, $$\frac{1 + x}{2 + 3x} = A + Bx + Cx^2 + Dx^3 + Ex^4.$$

Clearing, $$\begin{aligned} 1 + x = 2A + {} & 2Bx + 2Cx^2 + 2Dx^3 + 2Ex^4 \\ & + 3Ax + 3Bx^2 + 3Cx^3 + 3Dx^4 \end{aligned}$$

Equating coefficients of like powers of x,

$$2A = 1;\ 2B + 3A = 1;\ 2C + 3B = 0;\ 2D + 3C = 0;\ 2E + 3D = 0;$$

whence, $A = \frac{1}{2};\ B = -\frac{1}{4};\ C = \frac{3}{8};\ D = -\frac{9}{16};\ E = \frac{27}{32}.$

$$\therefore \frac{1 + x}{2 + 3x} = \frac{1}{2} - \frac{1}{4}x + \frac{3}{8}x^2 - \frac{9}{16}x^3 + \frac{27}{32}x^4.$$

6. Assume, $$\frac{1 + 2x}{1 - x - x^2} = A + Bx + Cx^2 + Dx^3 + Ex^4.$$

Clearing, $$\begin{aligned} 1 + 2x = A + {} & Bx + Cx^2 + Dx^3 + Ex^4 \\ & - Ax - Bx^2 - Cx^3 - Dx^4 \\ & \qquad - Ax^2 - Bx^3 - Cx^4. \end{aligned}$$

Equating coefficients of like powers of x,

$$A = 1;\ B - A = 2;\ C - B - A = 0;\ D - C - B = 0;\ E - D - C = 0;$$

whence, $A = 1;\ B = 3;\ C = 4;\ D = 7;\ E = 11.$

$$\therefore \frac{1 + 2x}{1 - x - x^2} = 1 + 3x + 4x^2 + 7x^3 + 11x^4.$$

7. Assume, $$\frac{1 - x}{1 - 3x - 2x^2} = A + Bx + Cx^2 + Dx^3 + Ex^4.$$

Clearing, $$\begin{aligned} 1 - x = A + {} & Bx + Cx^2 + Dx^3 + Ex^4 \\ & - 3Ax - 3Bx^2 - 3Cx^3 - 3Dx^4 \\ & \qquad - 2Ax^2 - 2Bx^3 - 2Cx^4. \end{aligned}$$

Equating coefficients of like powers of x,

$$A = 1;\ B - 3A = -1;\ C - 3B - 2A = 0;$$

$$D - 3C - 2B = 0;\ E - 3D - 2C = 0;$$

whence, $A = 1;\ B = 2;\ C = 8;\ D = 28;\ E = 100.$

$$\therefore \frac{1 - x}{1 - 3x - 2x^2} = 1 + 2x + 8x^2 + 28x^3 + 100x^4.$$

8. Assume, $\dfrac{2 - 3x}{1 + x + x^2} = A + Bx + Cx^2 + Dx^3 + Ex^4.$

Clearing,

$$\begin{aligned} 2 - 3x = A + Bx + Cx^2 &+ Dx^3 + Ex^4 \\ + Ax + Bx^2 &+ Cx^3 + Dx^4 \\ + Ax^2 &+ Bx^3 + Cx^4. \end{aligned}$$

Equating coefficients of like powers of x,

$$A=2;\ B+A=-3;\ C+B+A=0;\ D+C+B=0;\ E+D+C=0;$$

whence, $A = 2;\ B = -5;\ C = 3;\ D = 2;\ E = -5.$

$$\therefore \frac{2 - 3x}{1 + x + x^2} = 2 - 5x + 3x^2 + 2x^3 - 5x^4.$$

9. Assume, $\dfrac{1}{1 - 3x + 2x^2} = A + Bx + Cx^2 + Dx^3 + Ex^4.$

Clearing,

$$\begin{aligned} 1 = A + Bx + Cx^2 &+ Dx^3 + Ex^4 \\ - 3Ax - 3Bx^2 &- 3Cx^3 - 3Dx^4 \\ + 2Ax^2 &+ 2Bx^3 + 2Cx^4. \end{aligned}$$

Equating coefficients of like powers of x,

$$A = 1;\ B - 3A = 0;\ C - 3B + 2A = 0;$$

$$D - 3C + 2B = 0;\ E - 3D + 2C = 0;$$

whence, $A = 1;\ B = 3;\ C = 7;\ D = 15;\ E = 31.$

$$\therefore \frac{1}{1 - 3x + 2x^2} = 1 + 3x + 7x^2 + 15x^3 + 31x^4.$$

10. Assume, $\dfrac{3 - 2x}{1 - x + x^2} = A + Bx + Cx^2 + Dx^3 + Ex^4.$

Clearing,

$$\begin{aligned} 3 - 2x = A + Bx + Cx^2 &+ Dx^3 + Ex^4 \\ - Ax - Bx^2 &- Cx^3 - Dx^4 \\ + Ax^2 &+ Bx^3 + Cx^4. \end{aligned}$$

Equating coefficients of like powers of x,

$$A = 3;\ B - A = -2;\ C - B + A = 0;$$

$$D - C + B = 0;\ E - D + C = 0;$$

whence, $A = 3;\ B = 1;\ C = -2;\ D = -3;\ E = -1.$

$$\therefore \frac{3 - 2x}{1 - x + x^2} = 3 + x - 2x^2 - 3x^3 - x^4.$$

11. Assume, $\dfrac{x}{1+x+x^2} = A + Bx + Cx^2 + Dx^3 + Ex^4.$

Clearing,

$$\begin{aligned} x = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 \\ + Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 \\ + Ax^2 + Bx^3 + Cx^4 + Dx^5. \end{aligned}$$

Equating coefficients of like powers of x,

$$A = 0;\ B + A = 1;\ C + B + A = 0;\ D + C + B = 0;$$
$$E + D + C = 0;\ F + E + D = 0;$$

whence, $A = 0;\ B = 1;\ C = -1;\ D = 0;\ E = 1;\ F = -1.$

$$\therefore\ \frac{x}{1+x+x^2} = x - x^2 + 0\,x^3 + x^4 - x^5.$$

12. Assume, $\dfrac{2-x}{1-2x+x^2} = A + Bx + Cx^2 + Dx^3 + Ex^4.$

Clearing,

$$\begin{aligned} 2 - x = A + Bx + Cx^2 + Dx^3 + Ex^4 \\ - 2\,Ax - 2\,Bx^2 - 2\,Cx^3 - 2\,Dx^4 \\ + Ax^2 + Bx^3 + Cx^4. \end{aligned}$$

Equating coefficients of like powers of x,

$$A = 2;\ B - 2\cdot A = -1;\ C - 2\,B + A = 0;$$
$$D - 2\,C + B = 0;\ E - 2\,D + C = 0;$$

whence, $A = 2;\ B = 3;\ C = 4;\ D = 5;\ E = 6.$

$$\therefore\ \frac{2-x}{1-2x+x^2} = 2 + 3\,x + 4\,x^2 + 5\,x^3 + 6\,x^4.$$

13. Assume, $\dfrac{1}{1+2\,x^2+3\,x^4} = A + Bx^2 + Cx^4 + Dx^6 + Ex^8.$

Clearing,

$$\begin{aligned} 1 = A + Bx^2 + Cx^4 + Dx^6 + Ex^8 \\ + 2\,Ax^2 + 2\,Bx^4 + 2\,Cx^6 + 2\,Dx^8 \\ + 3\,Ax^4 + 3\,Bx^6 + 3\,Cx^8. \end{aligned}$$

Equating coefficients of like powers of x,

$$A = 1;\ B + 2\,A = 0;\ C + 2\,B + 3\,A = 0;$$
$$D + 2\,C + 3\,B = 0;\ E + 2\,D + 3\,C = 0;$$

whence, $A = 1;\ B = -2;\ C = 1;\ D = 4;\ E = -11.$

$$\therefore\ \frac{1}{1+2\,x^2+3\,x^4} = 1 - 2\,x^2 + x^4 + 4\,x^6 - 11\,x^8.$$

14. Assume, $\dfrac{x+5\,x^2}{1-4\,x+4\,x^2} = Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5.$

Clearing,

$$\begin{aligned} x + 5\,x^2 = Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 \\ -4\,Ax^2 - 4\,Bx^3 - 4\,Cx^4 - 4\,Dx^5 \\ +4\,Ax^3 + 4\,Bx^4 + 4\,Cx^5. \end{aligned}$$

Equating coefficients of like powers of x,

$$A=1;\ B-4A=5;\ C-4B+4A=0;$$
$$D-4C+4B=0;\ E-4D+4C=0;$$

whence, $A=1;\ B=9;\ C=32;\ D=92;\ E=240.$

$$\therefore\ \frac{x+5x^2}{1-4x+4x^2}=x+9x^2+32x^3+92x^4+240x^5.$$

PARTIAL FRACTIONS.

Page 341.

5. Assume, $$\frac{3x-3}{x^2-x-2}=\frac{A}{x-2}+\frac{B}{x+1}.$$

Clearing, $$3x-3=Ax+A+Bx-2B;$$
$$=(A+B)x+A-2B.$$

By principle, Art. 429, $A+B=3;\ A-2B=-3;$

whence, $A=1;\ B=2.$

$$\therefore\ \frac{3x-3}{x^2-x-2}=\frac{1}{x-2}+\frac{2}{x+1}.$$

6. Assume, $$\frac{5x+7}{x^2+3x+2}=\frac{A}{x+2}+\frac{B}{x+1}.$$

Clearing, $$5x+7=Ax+A+Bx+2B;$$
$$=(A+B)x+A+2B.$$

By principle, Art. 429, $A+B=5;\ A+2B=7;$

whence, $B=2;\ A=3.$

$$\therefore\ \frac{5x+7}{x^2+3x+2}=\frac{3}{x+2}+\frac{2}{x+1}.$$

7. Assume, $$\frac{2x-5}{x^2-5x+6}=\frac{A}{x-3}+\frac{B}{x-2}.$$

Clearing, $$2x-5=Ax-2A+Bx-3B;$$
$$=(A+B)x-2A-3B.$$

By principle, Art. 429, $A+B=2;\ -2A-3B=-5;$

whence, $A=1;\ B=1.$

$$\therefore\ \frac{2x-5}{x^2-5x+6}=\frac{1}{x-3}+\frac{1}{x-2}.$$

8. Assume, $\dfrac{7x+4}{x^2+2x-8}=\dfrac{A}{x+4}+\dfrac{B}{x-2}.$

Clearing, $7x+4=Ax-2A+Bx+4B;$

$=(A+B)x-2A+4B.$

By principle, Art. 429, $A+B=7;\ 4B-2A=4;$

whence, $A=4;\ B=3.$

$\therefore\ \dfrac{7x+4}{x^2+2x-8}=\dfrac{4}{x+4}+\dfrac{3}{x-2}.$

9. Assume, $\dfrac{9x-35}{x^2-8x+15}=\dfrac{A}{x-5}+\dfrac{B}{x-3}.$

Clearing, $9x-35=Ax-3A+Bx-5B;$

$=(A+B)x-3A-5B.$

By principle, Art. 429, $A+B=9;\ -3A-5B=-35;$

whence, $A=5;\ B=4.$

$\therefore\ \dfrac{9x-35}{x^2-8x+15}=\dfrac{5}{x-5}+\dfrac{4}{x-3}.$

10. Assume, $\dfrac{11x+13}{2x^2-7x+3}=\dfrac{A}{x-3}+\dfrac{B}{2x-1}.$

Clearing, $11x+13=2Ax-A+Bx-3B;$

$=(2A+B)x-A-3B.$

By principle, Art. 429, $2A+B=11;\ -A-3B=13;$

whence, $A=\frac{46}{5};\ B=-\frac{37}{5}.$

$$\therefore\ \frac{11x+13}{2x^2-7x+3}=\frac{\frac{46}{5}}{x-3}+\frac{\frac{-37}{5}}{2x-1}=\frac{46}{5(x-3)}-\frac{37}{5(2x-1)}.$$

11. Assume, $\dfrac{5x^2-2}{x^3-x^2-2x}=\dfrac{A}{x}+\dfrac{B}{x-2}+\dfrac{C}{x+1}.$

Clearing, $5x^2-2=Ax^2-Ax-2A+Bx^2+Bx+Cx^2-2Cx$

$=(A+B+C)x^2-(A-B+2C)x-2A.$

By principle, Art. 429, $A+B+C=5;\ -A+B-2C=0;\ -2A=-2.$

whence, $A=1;\ B=3;\ C=1.$

$\therefore\ \dfrac{5x^2-2}{x^3-x^2-2x}=\dfrac{1}{x}+\dfrac{3}{x-2}+\dfrac{1}{x+1}.$

12. Assume, $\dfrac{3x^2-6x+2}{x^3-3x^2+2x}=\dfrac{A}{x}+\dfrac{B}{x-2}+\dfrac{C}{x-1}.$

Clearing, $3x^2-6x+2=Ax^2-3Ax+2A+Bx^2-Bx+Cx^2-2Cx;$

$=(A+B+C)x^2-(3A+B+2C)x+2A.$

By principle, Art. 429, $A+B+C=3$; $-3A-B-2C=-6$; $2A=2$;

whence, $A=1$; $B=1$; $C=1$.

$$\therefore \frac{3x^2-6x+2}{x^3-3x^2+2x}=\frac{1}{x}+\frac{1}{x-2}+\frac{1}{x-1}.$$

13. Assume, $$\frac{3x+1}{(x-2)^2}=\frac{A}{(x-2)^2}+\frac{B}{x-2}.$$

Clearing, $3x+1=A+Bx-2B.$

By principle, Art. 429, $B=3$; $A-2B=1$;

whence, $B=3$; $A=7$.

$$\therefore \frac{3x+1}{(x-2)^2}=\frac{7}{(x-2)^2}+\frac{3}{x-2}.$$

14. Assume, $$\frac{x+5x^2}{(x-1)^3}=\frac{A}{(x-1)^3}+\frac{B}{(x-1)^2}+\frac{C}{x-1}.$$

Clearing, $x+5x^2=A+Bx-B+Cx^2-2Cx+C$;

$= Cx^2+(B-2C)x+A-B+C.$

By principle, Art. 429, $C=5$; $B-2C=1$; $A-B+C=0$;

whence, $A=6$; $B=11$; $C=5$.

$$\therefore \frac{x+5x^2}{(x-1)^3}=\frac{6}{(x-1)^3}+\frac{11}{(x-1)^2}+\frac{5}{x-1}.$$

REVERSION OF SERIES.

Page 343.

2. In this series, $a=1,\ b=1,\ c=1,\ d=1.$

Substituting in the formula under Art. 434,

$$x=y-y^2+y^3-y^4+\cdots.$$

3. In this series, $a=1,\ b=3,\ c=5,\ d=7.$

Substituting in the formula, $x=y-3y^2+13y^3-67y^4+\cdots.$

4. In this series, $a=1,\ b=-\frac{1}{2},\ c=\frac{1}{3},\ d=-\frac{1}{4}.$

Substituting in the formula, $x=y+\frac{1}{2}y^2+\frac{1}{6}y^3+\frac{1}{24}y^4+\cdots.$

5. In this series, $a=2,\ b=3,\ c=4,\ d=5.$

Substituting in the formula, $x=\frac{1}{2}y-\frac{3}{8}y^2+\frac{5}{16}y^3-\frac{35}{128}y^4+\cdots.$

6. In this series, $a=2,\ b=4,\ c=6,\ d=8.$

Substituting in the formula, $x=\frac{1}{2}y-\frac{1}{2}y^2+\frac{5}{8}y^3-\frac{7}{8}y^4+\cdots.$

Page 344.

8. Put y for $\frac{1}{2}$;

then, $y = x + \frac{x^2}{6} + \frac{x^3}{12} + \frac{x^4}{20} + \cdots$.

Reverting, $x = y - \frac{1}{6}y^2 - \frac{1}{36}y^3 - \frac{1}{270}y^4 + \cdots$.

Restoring value of y, $x = \frac{1}{2} - \frac{1}{24} - \frac{1}{288} - \frac{1}{4320}$

$= .5 - .041667 - .003472 - .000231$

$= .45463.$

9. Put y for $\frac{1}{4}$;

then, $y = x - \frac{3}{8}x^2 + \frac{5}{24}x^3 - \frac{7}{48}x^4 + \cdots$.

Reverting, $x = y + \frac{3}{8}y^2 + \frac{7}{96}y^2 + \frac{29}{1536}y^4 + \cdots$.

Restoring value of y, $x = \frac{1}{4} + \frac{3}{128} + \frac{7}{6144} + \frac{29}{393216}$

$= .25 + .023437 + .001139 + .000073$

$= .274649+.$

RECURRING SERIES.

Page 345.

2. Assume m, n to be the scale of relation;

then, $3x^2 = m + 2nx$;

$5x^3 = 2mx + 3nx^2$.

Solving, $m = x^2$; $n = x$.

$\therefore$ x^2, x is the scale of relation.

3. Assume m, n to be the scale of relation;

then, $5x^2 = m + 3nx$;

$11x^3 = 3mx + 5nx^2$.

Solving, $m = 2x^2$; $n = x$.

$\therefore$ $2x^2$, x is the scale of relation.

4. Assume m, n to be the scale of relation;

then, $17x^2 = m + 7nx$;

$55x^3 = 7mx + 17nx^2$.

Solving, $m = 3x^2$; $n = 2x$.

$\therefore$ $3x^2$, $2x$ is the scale of relation.

5. Assume m, n to be the scale of relation;

then, $8x^2 = m + 2nx$;

$28x^3 = 2mx + 8nx^2$.

Solving, $m = 2x^2$; $n = 3x$.

$\therefore$ $2x^2$, $3x$ is the scale of relation.

Page 347.

2. The scale of relation is $m = x^2$.

$$n = x;\ a = 1;\ b = 3x.$$

Substituting in formula (2), Art. 439,

$$s = \frac{1 + 3x - x}{1 - x^2 - x} = \frac{1 + 2x}{1 - x - x^2}.$$

3. The scale of relation is $m = 2x^2$.

$$n = x;\ a = 1;\ b = 3x.$$

Substituting in formula (2), Art. 439,

$$s = \frac{1 + 3x - x}{1 - 2x^2 - x} = \frac{1 + 2x}{1 - x - 2x^2}.$$

4. The scale of relation is $m = 2x^2$.

$$n = 3x;\ a = 1;\ b = 2x.$$

Substituting in formula (2), Art. 439,

$$s = \frac{1 + 2x - 3x}{1 - 2x^2 - 3x} = \frac{1 - x}{1 - 3x - 2x^2}.$$

5. The scale of relation is $m = -x^2$.

$$n = 2x;\ a = 1;\ b = 3x.$$

Substituting in formula (2), Art. 439,

$$s = \frac{1 + 3x - 2x}{1 + x^2 - 2x} = \frac{1 + x}{1 - 2x + x^2}.$$

BINOMIAL THEOREM.

Page 351.

4. $$(a + b)^{-2} = a^{-2} + (-2)a^{-3}b + \left(\frac{-2 \cdot -3}{1 \cdot 2}\right)a^{-4}b^2 + \left(\frac{-2 \cdot -3 \cdot -4}{1 \cdot 2 \cdot 3}\right)a^{-5}b^3 + \cdots$$

$$= a^{-2} - 2a^{-3}b + 3a^{-4}b^2 - 4a^{-5}b^3 + \cdots.$$

5. $$(a + b)^{-\frac{1}{2}} = a^{-\frac{1}{2}} + (-\tfrac{1}{2})a^{-\frac{3}{2}}b + \left(\frac{-\frac{1}{2} \cdot -\frac{3}{2}}{1 \cdot 2}\right)a^{-\frac{5}{2}}b^2 + \left(\frac{-\frac{1}{2} \cdot -\frac{3}{2} \cdot -\frac{5}{2}}{1 \cdot 2 \cdot 3}\right)a^{-\frac{7}{2}}b^3 + \cdots$$

$$= a^{-\frac{1}{2}} - \tfrac{1}{2}a^{-\frac{3}{2}}b + \tfrac{3}{8}a^{-\frac{5}{2}}b^2 - \tfrac{5}{16}a^{-\frac{7}{2}}b^3 + \cdots.$$

6. $(a-b)^{-3} = a^{-3} - (-3)a^{-4}b + \left(\frac{-3 \cdot -4}{1 \cdot 2}\right)a^{-5}b^2$
$$-\left(\frac{-3 \cdot -4 \cdot -5}{1 \cdot 2 \cdot 3}\right)a^{-6}b^3 + \cdots$$
$$= a^{-3} + 3\,a^{-4}b + 6\,a^{-5}b^2 + 10\,a^{-6}b^3 + \cdots.$$

7. $(a-b)^{\frac{3}{4}} = a^{\frac{3}{4}} - (\tfrac{3}{4})a^{-\frac{1}{4}}b + \left(\frac{\frac{3}{4} \cdot -\frac{1}{4}}{1 \cdot 2}\right)a^{-\frac{5}{4}}b^2 - \left(\frac{\frac{3}{4} \cdot -\frac{1}{4} \cdot -\frac{5}{4}}{1 \cdot 2 \cdot 3}\right)a^{-\frac{9}{4}}b^3 + \cdots$
$$= a^{\frac{3}{4}} - \tfrac{3}{4}\,a^{-\frac{1}{4}}b - \tfrac{3}{32}\,a^{-\frac{5}{4}}b^2 - \tfrac{15}{384}\,a^{-\frac{9}{4}}b^3 + \cdots.$$

8. $(a+b)^{-\frac{1}{4}} = a^{-\frac{1}{4}} + (-\tfrac{1}{4})a^{-\frac{5}{4}}b + \left(\frac{-\frac{1}{4} \cdot -\frac{5}{4}}{1 \cdot 2}\right)a^{-\frac{9}{4}}b^2$
$$+\left(\frac{-\frac{1}{4} \cdot -\frac{5}{4} \cdot -\frac{9}{4}}{1 \cdot 2 \cdot 3}\right)a^{-\frac{13}{4}}b^3 + \cdots$$
$$= a^{-\frac{1}{4}} - \tfrac{1}{4}\,a^{-\frac{5}{4}}b + \tfrac{5}{32}\,a^{-\frac{9}{4}}b^2 - \tfrac{15}{128}\,a^{-\frac{13}{4}}b^3 + \cdots.$$

9. $(a+b)^{-\frac{3}{2}} = a^{-\frac{3}{2}} + (-\tfrac{3}{2})a^{-\frac{5}{2}}b + \left(\frac{-\frac{3}{2} \cdot -\frac{5}{2}}{1 \cdot 2}\right)a^{-\frac{7}{2}}b^2$
$$+\left(\frac{-\frac{3}{2} \cdot -\frac{5}{2} \cdot -\frac{7}{2}}{1 \cdot 2 \cdot 3}\right)a^{-\frac{9}{2}}b^3 + \cdots$$
$$= a^{-\frac{3}{2}} - \tfrac{3}{2}\,a^{-\frac{5}{2}}b + \tfrac{15}{8}\,a^{-\frac{7}{2}}b^2 - \tfrac{35}{16}\,a^{-\frac{9}{2}}b^3 + \cdots.$$

10. $(1+3\,x)^{\frac{5}{3}} = 1^{\frac{5}{3}} + \tfrac{5}{3}(1)^{\frac{2}{3}}(3\,x) + \left(\frac{\frac{5}{3} \cdot \frac{2}{3}}{1 \cdot 2}\right)(1)^{-\frac{1}{3}}(3\,x)^2$
$$+\left(\frac{\frac{5}{3} \cdot \frac{2}{3} \cdot -\frac{1}{3}}{1 \cdot 2 \cdot 3}\right)(1)^{-\frac{4}{3}}(3\,x)^3 + \cdots$$
$$= 1 + 5\,x + 5\,x^2 - \tfrac{5}{3}\,x^3 + \cdots.$$

11. $(2\,a+3\,b)^{\frac{3}{4}} = (2\,a)^{\frac{3}{4}} + \tfrac{3}{4}(2\,a)^{-\frac{1}{4}}(3\,b) + \left(\frac{\frac{3}{4} \cdot -\frac{1}{4}}{1 \cdot 2}\right)(2\,a)^{-\frac{5}{4}}(3\,b)^2$
$$+\left(\frac{\frac{3}{4} \cdot -\frac{1}{4} \cdot -\frac{5}{4}}{1 \cdot 2 \cdot 3}\right)(2\,a)^{-\frac{9}{4}}(3\,b)^3 + \cdots$$
$$= \sqrt[4]{8\,a^3} + \frac{9\,b}{4\sqrt[4]{2\,a}} - \frac{27\,b^2}{64\,a\sqrt[4]{2\,a}} + \frac{135\,b^3}{512\,a^2\sqrt[4]{2\,a}} + \cdots.$$

12. $(1-2\,x)^{\frac{1}{4}} = (1)^{\frac{1}{4}} - \tfrac{1}{4}(1)^{-\frac{3}{4}}(2\,x) + \left(\frac{\frac{1}{4} \cdot -\frac{3}{4}}{1 \cdot 2}\right)(1)^{-\frac{7}{4}}(2\,x)^2$
$$-\left(\frac{\frac{1}{4} \cdot -\frac{3}{4} \cdot -\frac{7}{4}}{1 \cdot 2 \cdot 3}\right)(1)^{-\frac{11}{4}}(2\,x)^3 + \cdots$$
$$= 1 - \tfrac{1}{2}\,x - \tfrac{3}{8}\,x^2 - \tfrac{7}{16}\,x^3 - \cdots.$$

13. $$\frac{1}{\sqrt[5]{(a-b)^3}}=(a-b)^{-\frac{3}{5}}=a^{-\frac{3}{5}}-(-\tfrac{3}{5})a^{-\frac{8}{5}}b+\left(\frac{-\frac{3}{5}\cdot-\frac{8}{5}}{1\cdot2}\right)a^{-\frac{13}{5}}b^2$$
$$-\left(\frac{-\frac{3}{5}\cdot-\frac{8}{5}\cdot-\frac{13}{5}}{1\cdot2\cdot3}\right)a^{-\frac{18}{5}}b^3+\cdots$$
$$=a^{-\frac{3}{5}}+\tfrac{3}{5}a^{-\frac{8}{5}}b+\tfrac{12}{25}a^{-\frac{13}{5}}b^2+\tfrac{52}{125}a^{-\frac{18}{5}}b^3+\cdots.$$

15. $$\sqrt{7}=\sqrt{2^2+3}=(2^2+3)^{\frac{1}{2}}=(2^2)^{\frac{1}{2}}+\tfrac{1}{2}(2^2)^{-\frac{1}{2}}.3-\tfrac{1}{8}(2^2)^{-\frac{3}{2}}.3^2+\cdots$$
$$=2+\frac{1\times3}{2\times2}-\frac{1\times9}{8\times8}$$
$$=2+.75-.1406=2.6094.$$

16. $$\sqrt{18}=\sqrt{4^2+2}=(4^2+2)^{\frac{1}{2}}=(4^2)^{\frac{1}{2}}+\tfrac{1}{2}(4^2)^{-\frac{1}{2}}.2-\tfrac{1}{8}(4^2)^{-\frac{3}{2}}.2^2+\cdots$$
$$=4+\frac{1\times2}{2\times4}-\frac{1\times4}{8\times64}$$
$$=4+.25-.00781=4.2421.$$

17. $$\sqrt{26}=\sqrt{5^2+1}=(5^2+1)^{\frac{1}{2}}=(5^2)^{\frac{1}{2}}+\tfrac{1}{2}(5^2)^{-\frac{1}{2}}.1-\tfrac{1}{8}(5^2)^{-\frac{3}{2}}.1^2+\cdots$$
$$=5+\frac{1\times1}{2\times5}-\frac{1\times1}{8\times125}$$
$$=5+.1-.001=5.099+.$$

18. $$\sqrt[3]{29}=\sqrt[3]{3^3+2}=(3^3+2)^{\frac{1}{3}}=(3^3)^{\frac{1}{3}}+\tfrac{1}{3}(3^3)^{-\frac{2}{3}}.2-\tfrac{1}{9}(3^3)^{-\frac{5}{3}}.2^2+\cdots$$
$$=3+\frac{1\times2}{3\times9}-\frac{1\times4}{9\times243}$$
$$=3+.07407-.00182=3.07225.$$

19. $$\sqrt[4]{85}=\sqrt[4]{3^4+4}=(3^4+4)^{\frac{1}{4}}=(3^4)^{\frac{1}{4}}+\tfrac{1}{4}(3^4)^{-\frac{3}{4}}.4-\tfrac{3}{32}(3^4)^{-\frac{7}{4}}.4^2+\cdots$$
$$=3+\frac{1\times4}{4\times27}-\frac{3\times16}{32\times2187}$$
$$=3+.03703-.00068\doteq3.03635.$$

20. $$\sqrt[5]{245}=\sqrt[5]{3^5+2}=(3^5+2)^{\frac{1}{5}}=(3^5)^{\frac{1}{5}}+\tfrac{1}{5}(3^5)^{-\frac{4}{5}}.2-\tfrac{2}{25}(3^5)^{-\frac{9}{5}}.2^2+\cdots$$
$$=3+\frac{1\times2}{5\times81}-\frac{2\times4}{25\times59049}$$
$$=3+.00493-.000005=3.00492.$$

THEORY OF EQUATIONS.

Page 354.

2. If 2 is a root, $x - 2$ is a factor, and the equation may be written,

$$(x-2)(x^2-2x-3)=0;$$

whence, $x^2-2x-3=0;$

and, $x^2-2x=3.$

Solving, $x=3$, and -1, the other roots.

3. If -1 is a root, $x+1$ is a factor, and the equation may be written,

$$(x+1)(x^2-3x+2)=0;$$

whence, $x^2-3x+2=0;$

and, $x^2-3x=-2.$

Solving, $x=2$, and 1, the other roots.

4. If 4 is a root, $x-4$ is a factor, and the equation may be written,

$$(x-4)(x^2+x-6)=0;$$

whence, $x^2+x-6=0;$

and, $x^2+x=6.$

Solving, $x=-3$, and 2, the other roots.

5. If 5 is a root, $x-5$ is a factor, and the equation may be written,

$$(x-5)(x^2-10x+16)=0;$$

whence, $x^2-10x+16=0;$

and, $x^2-10x=-16.$

Solving, $x=8$, and 2, the other roots.

6. If -2 and 3 are roots, $x+2$ and $x-3$ are factors, and the equation may be written,

$$(x+2)(x-3)(x^2+x+10)=0;$$

whence, $x^2+x+10=0;$

and, $x^2+x=-10.$

Solving, $x=-\frac{1}{2}\pm\frac{1}{2}\sqrt{-39}$, the other roots.

Page 356.

1. The equation is, $(x-2)(x-3)(x-4)=0;$

which becomes, $x^3-9x^2+26x-24=0.$

2. The equation is, $(x-3)(x-1)(x-5)=0;$

which becomes, $x^3-9x^2+23x-15=0.$

3. The equation is, $(x+1)(x+3)(x-2)=0;$

which becomes, $x^3+2x^2-5x-6=0.$

4. The equation is, $(x-1)(x-2)(x-3)(x+4)=0;$

which becomes, $x^4-2x^3-13x^2+38x-24=0.$

5. The equation is, $(x-1)(x-3)(x-\frac{1}{2})=0;$

which becomes, $x^3-4\frac{1}{2}x^2+5x-1\frac{1}{2}=0.$

6. The equation is, $(x-2)(x+\frac{1}{2})(x+3)(x-\frac{1}{3})=0;$

which becomes, $x^4+\frac{7}{6}x^3-6x^2-\frac{7}{6}x+1=0.$

7. If 1 and 2 are roots, $x-1$ and $x-2$ are factors, and the equation may be written,

$$(x-1)(x-2)(x^2-7x+12)=0;$$

whence, $x^2-7x+12=0;$

and, $x^2-7x=-12.$

Solving, $x=4$, or 3.

$\therefore$ 1, 2, 3, 4 are the roots.

8. If 2 and -1 are roots, $y-2$ and $y+1$ are factors, and the equation may be written,

$$(y-2)(y+1)(y^2-4y-4)=0;$$

whence, $(y^2-4y-4)=0;$

and, $y^2-4y=4.$

Solving, $y=2\pm2\sqrt{2}.$

$\therefore$ -1, 2, and $2\pm2\sqrt{2}$ are the roots.

TRANSFORMATION OF EQUATIONS.

Page 359.

2. Supplying the missing term,

$$x^5+3x^4\pm0x^3-2x^2+3x-6=0.$$

Put $-y$ for x, $y^5-3y^4\pm0y^3+2y^2+3y+6=0$

$$=y^5-3y^4+2y^2+3y+6=0.$$

3. Supplying the missing terms,

$$x^6\pm0x^5-3x^4\pm0x^3-4x^2+7x+13=0.$$

Put $-y$ for x, $y^6\mp0y^5-3y^4\mp0y^3-4y^2-7y+13=0$

$$=y^6-3y^4-4y^2-7y+13=0.$$

4. Supplying the missing term,

$$x^5 + 2x^4 - 5x^3 \pm 0x^2 - 3x + 4 = 0.$$

Put $-y$ for x.

$$y^5 - 2y^4 - 5y^3 \mp 0y^2 - 3y - 4 = 0$$
$$= y^5 - 2y^4 - 5y^3 - 3y - 4 = 0.$$

1. Supplying the missing term, $x^3 \pm 0x^2 + 4x - 16 = 0.$

Multiplying the first term by 1, the second by 3, the third by 9, etc.,

$$y^3 \pm 0y^2 + 36y - 432 = 0$$
$$= y^3 + 36y - 432 = 0.$$

2. Supplying the missing term, $x^3 + 2x^2 \pm 0x - 3 = 0.$

Multiplying the first term by 1, the second by 3, the third by 9, etc.,

$$y^3 + 6y^2 \pm 0y - 81 = 0$$
$$= y^3 + 6y^2 - 81 = 0.$$

3. Supplying the missing term, $x^3 - 2x^2 \pm 0x - 9 = 0.$

Multiplying the first term by 1, the second by 3, the third by 9, etc.

$$y^3 - 6y^2 \pm 0y - 243 = 0$$
$$= y^3 - 6y^2 - 243 = 0.$$

4. Supplying the missing term, $x^3 \pm 0x^2 - 3x - 2 = 0.$

Multiplying the first term by 1, the second by 3, the third by 9, etc.

$$y^3 \pm 0y^2 - 27y - 54 = 0$$
$$= y^3 - 27y - 54 = 0.$$

Page 360.

1. Putting $\frac{y}{2}$ for x, the equation becomes,

$$\frac{y^4}{32} - \frac{y^3}{32} + \frac{y}{16} - 9 = 0.$$

Clearing, $y^4 - y^3 + 2y - 288 = 0.$

2. Putting $\frac{y}{6}$ for x, the equation becomes,

$$\frac{y^3}{216} - \frac{7y^2}{108} + \frac{11y}{216} - \frac{25}{72} = 0.$$

Clearing, $y^3 - 14y^2 + 11y - 75 = 0.$

3. Putting $\frac{y}{6}$ for x, the equation becomes,

$$\frac{y^3}{216} - \frac{y^2}{12} + \frac{3y}{24} - \frac{5}{9} = 0.$$

Clearing, $y^3 - 18y^2 + 27y - 120 = 0$

4. Putting $\frac{y}{2}$ for x, the equation becomes,

$$\frac{y^3}{8}-\frac{11y^2}{8}-\frac{3y}{2}+\frac{31}{8}=0.$$

Clearing, $$y^3-11y^2-12y+31=0.$$

5. Putting $\frac{y}{15}$ for x, the equation becomes,

$$\frac{y^4}{50625}-\frac{4y^3}{16875}+\frac{2y^2}{675}-\frac{4y}{15}+\frac{7}{15}=0.$$

Clearing, $y^4-12y^3+150y^2-13500y+23625=0.$

6. Putting $\frac{y}{60}$ for x, the equation becomes,

$$\frac{y^4}{12960000}-\frac{13y^3}{2592000}+\frac{13y^2}{144000}-\frac{17y}{1440}+\frac{1}{80}=0.$$

Clearing, $y^4-65y^3+1170y^2-153000y+162000=0.$

7. Putting $\frac{y}{10}$ for x, the equation becomes,

$$\frac{y^4}{10000}+\frac{2y^3}{5000}-\frac{3y^2}{400}-\frac{y}{20}=0.$$

Clearing, $$y^4+4y^3-75y^2+500y=0.$$

8. Putting $\frac{y}{12}$ for x, the equation becomes,

$$\frac{y^5}{248832}-\frac{y^3}{6912}+\frac{y}{18}-24=0.$$

Clearing, $$y^5-36y^3+13824y-5971968=0.$$

Page 363.

1. $$\begin{array}{l} 1-14+71-154+120\,\underline{|-5} \\ \underline{\quad\;\; -5+45-130+120} \\ 1-\;\,9+26-\;\,24 \end{array}$$

$\therefore$ The quotient is $x^3-9x^2+26x-24$.

2. $$\begin{array}{l} 1+6-10-112-207-110\,\underline{|+2} \\ \underline{\quad +2+\;\,8-\;\,36-152-110} \\ 1+4-18-\;\,76-\;\,55 \end{array}$$

$\therefore$ The quotient is $x^4+4x^3-18x^2-76x-55$.

3. $$\begin{array}{l} 1-12+47-72+36\,\underline{|-6} \\ \underline{\quad\;\; -6+36-66+36} \\ 1-\;\,6+11-6 \end{array}$$

$\therefore$ The quotient is $x^3-6x^2+11x-6$.

Page 364.

3.
$$\begin{array}{ll}
1+8+21+25+\ 8\,\lfloor\underline{+2} & \\
\quad 2+12+18+14 & \\
\cline{1-1}
\quad 6+\ 9+\ 7-\ 6 & \therefore\quad -6=\text{first remainder.}\\
\quad 2+\ 8+\ 2 & \\
\cline{1-1}
\quad 4+\ 1+\ 5 & \therefore\quad +5=\text{second remainder.}\\
\quad 2+\ 4 & \\
\cline{1-1}
\quad 2-\ 3 & \therefore\quad -3=\text{third remainder.}\\
\quad 2 & \\
\cline{1-1}
\quad 0 & \therefore\quad 0=\text{fourth remainder.}
\end{array}$$

Since the remainders are the required coefficients, the equation becomes,

$$x^4-3x^2+5x-6=0.$$

4.
$$\begin{array}{ll}
1-15\pm\ \ 0+\ \ 9\,\lfloor\underline{+5} & \\
\quad +\ 5-100+500 & \\
\cline{1-1}
\quad -20+100-491 & \therefore\ -491=\text{first remainder.}\\
\qquad 5-125 & \\
\cline{1-1}
\quad -25+225 & \therefore\quad 225=\text{second remainder.}\\
\qquad 5 & \\
\cline{1-1}
\quad -30 & \therefore\quad -30=\text{third remainder.}
\end{array}$$

Since the remainders are the required coefficients, the equation becomes,

$$x^3-30x^2+225x-491=0.$$

5.
$$\begin{array}{ll}
1\pm\ 0+\ \ 5\pm\ \ 0-\ \ 3-\ \ 5\,\lfloor\underline{-10} & \\
\quad -10-100-1050-10500-104970 & \\
\cline{1-1}
\quad 10+105+1050+10497+104965 & \therefore\ 104965=\text{first remainder.}\\
\quad -10-200-3050-41000 & \\
\cline{1-1}
\quad 20+305+4100+51497 & \therefore\ 51497=\text{second remainder.}\\
\quad -10-300-6050 & \\
\cline{1-1}
\quad 30+605+10150 & \therefore\ 10150=\text{third remainder.}\\
\quad -10-400 & \\
\cline{1-1}
\quad 40-1005 & \therefore\ 1005=\text{fourth remainder.}\\
\quad -10 & \\
\cline{1-1}
\quad 50 & \therefore\quad 50=\text{fifth remainder.}
\end{array}$$

Since the remainders are the required coefficients, the equation becomes,

$$y^5+50y^4+1005y^3+10150y^2+51497y+104965=0.$$

6.
$$\begin{array}{ll}
1 - 3 - 15 + 49 - 12\,\underline{|-3} & \\
\underline{-3 - 0 + 45 - 12} & \\
0 - 15 + 4 - 0 & \therefore \quad 0 = \text{first remainder.}\\
\underline{-3 - 9 + 18} & \\
3 - 6 - 14 & \therefore \quad -14 = \text{second remainder.}\\
\underline{-3 - 18} & \\
6 + 12 & \therefore \quad 12 = \text{third remainder.}\\
\underline{-3} & \\
9 & \therefore \quad 9 = \text{fourth remainder.}
\end{array}$$

The equation becomes, $y^4 + 9y^3 + 12y^2 - 14y = 0.$

7.
$$\begin{array}{ll}
1 + 3 - 3 + 7 - 317\,\underline{|-10} & \\
\underline{-10 - 130 - 1270 - 12770} & \\
13 + 127 + 1277 + 12453 & \therefore \quad 12453 = \text{first remainder.}\\
\underline{-10 - 230 - 3570} & \\
23 + 357 + 4847 & \therefore \quad 4847 = \text{second remainder.}\\
\underline{-10 - 330} & \\
33 + 687 & \therefore \quad 687 = \text{third remainder.}\\
\underline{-10} & \\
43 & \therefore \quad 43 = \text{fourth remainder.}
\end{array}$$

The equation becomes,

$$y^4 + 43y^3 + 687y^2 + 4847y + 12453 = 0.$$

Page 365.

8.
$$\begin{array}{ll}
1 \pm 0 + 37 \pm 0 - 94 + 13\,\underline{|-2} & \\
\underline{-2 - 4 - 82 - 164 - 140} & \\
2 + 41 + 82 + 70 + 153 & \therefore \quad 153 = \text{first remainder.}\\
\underline{-2 - 8 - 98 - 360} & \\
4 + 49 + 180 + 430 & \therefore \quad 430 = \text{second remainder.}\\
\underline{-2 - 12 - 122} & \\
6 + 61 + 302 & \therefore \quad 302 = \text{third remainder.}\\
\underline{-2 - 16} & \\
8 + 77 & \therefore \quad 77 = \text{fourth remainder.}\\
\underline{-2} & \\
10 & \therefore \quad 10 = \text{fifth remainder.}
\end{array}$$

The equation becomes,

$$x^5 + 10x^4 + 77x^3 + 302x^2 + 430x + 153 = 0.$$

9.
$$\begin{array}{ll}
1 - 22 + 179 - 638 + 840 \,\lfloor\underline{-3} & \\
\quad -\ 3 + \ 57 - 366 + 816 & \\
\cline{1-1}
\quad - 19 + 122 - 272 + \ 24 & \therefore \quad 24 = \text{first remainder.} \\
\quad -\ 3 + \ 48 - 222 & \\
\cline{1-1}
\quad - 16 + \ 74 - \ 50 & \therefore \quad -50 = \text{second remainder.} \\
\quad -\ 3 + \ 39 & \\
\cline{1-1}
\quad - 13 + \ 35 & \therefore \quad 35 = \text{third remainder.} \\
\quad -\ 3 & \\
\cline{1-1}
\quad - 10 & \therefore \quad -10 = \text{fourth remainder.}
\end{array}$$

The equation becomes,

$$y^4 - 10y^3 + 35y^2 - 50y + 24 = 0.$$

10. (1.)
$$\begin{array}{ll}
1 - 4 + \ 3 - 2 + \ 1 - \ 1 \,\lfloor\underline{-2} & \\
\quad - 2 + \ 4 + 2 + \ 8 + 14 & \\
\cline{1-1}
\quad - 2 - \ 1 - 4 - \ 7 - 15 & \therefore \quad -15 = \text{first remainder.} \\
\quad - 2 - \ 0 + 2 + 12 & \\
\cline{1-1}
\quad\ \ 0 - \ 1 - 6 - 19 & \therefore \quad -19 = \text{second remainder.} \\
\quad - 2 - \ 4 - 6 & \\
\cline{1-1}
\quad\ \ 2 + \ 3 + 0 & \therefore \quad 0 = \text{third remainder.} \\
\quad - 2 - \ 8 & \\
\cline{1-1}
\quad\ \ 4 + 11 & \therefore \quad 11 = \text{fourth remainder.} \\
\quad - 2 & \\
\cline{1-1}
\quad\ \ 6 & \therefore \quad 6 = \text{fifth remainder.}
\end{array}$$

The equation becomes,

$$y^5 + 6y^4 + 11y^3 - 19y - 15 = 0.$$

10. (2.)
$$\begin{array}{ll}
1 + 40 - \ 20 \pm \ 0 + \ 35 \,\lfloor\underline{-2} & \\
\quad -\ 2 - \ 84 - 128 - 256 & \\
\cline{1-1}
\quad\ \ 42 + \ 64 + 128 + 291 & \therefore \quad 291 = \text{first remainder.} \\
\quad -\ 2 - \ 88 - 304 & \\
\cline{1-1}
\quad\ \ 44 + 152 + 432 & \therefore \quad 432 = \text{second remainder.} \\
\quad -\ 2 - \ 92 & \\
\cline{1-1}
\quad\ \ 46 + 244 & \therefore \quad 244 = \text{third remainder.} \\
\quad -\ 2 & \\
\cline{1-1}
\quad\ \ 48 & \therefore \quad 48 = \text{fourth remainder.}
\end{array}$$

The equation becomes,

$$y^4 + 48y^3 + 244y^2 + 432y + 291 = 0.$$

10. (3.)

$$\begin{array}{lll}
1 \pm 0 \pm 0 \pm 0 \pm 0 - 1 \,\underline{|-2} & & \\
\underline{\quad - 2 - 4 - 8 - 16 - 32} & & \\
\quad 2 + 4 + 8 + 16 + 31 & \therefore & 31 = \text{first remainder.} \\
\underline{\quad - 2 - 8 - 24 - 64} & & \\
\quad 4 + 12 + 32 + 80 & \therefore & 80 = \text{second remainder.} \\
\underline{\quad - 2 - 12 - 48} & & \\
\quad 6 - 24 + 80 & \therefore & 80 = \text{third remainder.} \\
\underline{\quad - 2 - 16} & & \\
\quad 8 + 40 & \therefore & 40 = \text{fourth remainder.} \\
\underline{\quad - 2} & & \\
\quad 10 & \therefore & 10 = \text{fifth remainder.}
\end{array}$$

The equation becomes,

$$x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 31 = 0.$$

10. (4.)

$$\begin{array}{lll}
1 \pm 0 - 80 + 90 \,\underline{|-2} & & \\
\underline{\quad -2 - 4 + 152} & & \\
\quad 2 - 76 - 62 & \therefore & -62 = \text{first remainder.} \\
\underline{\quad -2 - 8} & & \\
\quad 4 - 68 & \therefore & -68 = \text{second remainder.} \\
\underline{\quad -2} & & \\
\quad 6 & \therefore & 6 = \text{third remainder.}
\end{array}$$

The equation becomes, $y^3 + 6y^2 - 68y - 62 = 0.$

COMMENSURABLE ROOTS.

Page 366.

4. There are three roots, and they are all positive. The product of the roots is 20; therefore, try 5, which is one of the factors of 20.

$$\begin{array}{l}
1 - 10 + 29 - 20 \,\underline{|-5} \\
\underline{\quad - 5 + 25 - 20} \\
\quad - 5 + 4
\end{array}$$

$x^2 - 5x + 4 = 0$ is the equation which will give the other roots. Solving, x is found equal to 4 and 1; ∴ 5, 4, and 1 are the commensurable roots.

5. We know that there are three roots, one negative and two positive. Putting $\frac{y}{6}$ for x, the equation becomes, $y^3 - 11y^2 - 270y + 2592 = 0$. Try -16 as one root, since 16 is a factor of 2592.

$$\begin{array}{r|l}
1 - 11 - 270 + 2592 & \underline{+16} \\
\underline{+16 - 432 + 2592} & \\
-27 + 162 &
\end{array}$$

$y^2 - 27y + 162 = 0$ is the equation which will give the other roots. Solving, y is found to be equal to 18 and 9. Since $x = \frac{y}{6}$, the roots of the original equation are $\frac{3}{2}$, 3, $-\frac{8}{3}$.

6. We know that the equation has four roots, two positive and two negative, and their product is 60. Try 5, a factor of 60, as one of the roots.

$$\begin{array}{r|l}
1 + 5 - 28 - 122 + 60 & \underline{-5} \\
\underline{-5 - 50 - 110 + 60} & \\
10 + 22 - 12 &
\end{array}$$

Try -6 as another root,

$$\begin{array}{r|l}
10 + 22 - 12 & \underline{+6} \\
\underline{+6 + 24 - 12} & \\
4 - 2 &
\end{array}$$

Then $x^2 + 4x - 2 = 0$ is the equation which will give the other roots. Solving, x is found equal to $-2 \pm \sqrt{6}$. $\therefore$ 5, -6, $-\sqrt{6} - 2$, $\sqrt{6} - 2$, are the roots.

7. There are four roots, two negative and two positive, and their product is 192. Try 8, a factor of 192, as one root.

$$\begin{array}{lr|l}
 & 1 - 5 - 46 + 152 + 192 & \underline{-8} \\
 & \underline{-8 - 24 + 176 + 192} & \\
 & 3 - 22 - 24 & \underline{-4} \\
\text{Try 4 as another,} & \underline{-4 - 28 - 24} & \\
 & 7 + 6 &
\end{array}$$

Then, $x^2 + 7x + 6 = 0$ is the equation which will give the other roots. Solving, x is found equal to -6 and -1. $\therefore$ 4, 8, -6, -1 are the roots.

8. Putting $\frac{y}{6}$ for x, the equation becomes, $y^4 - 25y^3 + 186y^2 - 900y + 5400 = 0$. Try 10, a factor of 5400, as one root.

$$\begin{array}{lr|l}
 & 1 - 25 + 186 - 900 + 5400 & \underline{-10} \\
 & \underline{-10 + 150 - 360 + 5400} & \\
 & -15 + 36 - 540 & \underline{-15} \\
\text{Try 15 as another,} & \underline{-15 - 0 - 540} & \\
 & 0 + 36 &
\end{array}$$

Then, $x^2 + 36 = 0$ is the equation which will give the other roots. Solving, x is found equal to $\pm 6\sqrt{-1}$. Since $x = \frac{y}{6}$, the roots of the original equation are $\frac{10}{6}$, $\frac{15}{6}$, $\frac{\pm 6\sqrt{-1}}{6}$, or $\frac{5}{3}$, $\frac{5}{2}$, $\pm\sqrt{-1}$.

9. There are four roots, all positive, and their product is 168. Try 2, a factor of 168, as one root.

$$\begin{array}{l} 1-16+89-206+168\,\underline{|-2} \\ \underline{-\ \ 2+28-122+168} \\ -14+61-\ \ 84\,\underline{|-4} \\ \underline{-\ \ 4+40-\ \ 84} \\ -10+21 \end{array}$$

Try 4 as another,

Then, $x^2-10x+21=0$ is the equation which will give the other roots. Solving, x is found equal to 3 and 7. $\therefore$ 2, 3, 4, 7 are the roots.

10. There are 4 roots, all positive, and their product is 15876. Try 7, a factor of 15876, as one root.

$$\begin{array}{l} 1-48+827-6048+15876\,\underline{|-7} \\ \underline{-\ \ 7+287-3780+15876} \\ -41+540-2268\,\underline{|-9} \\ \underline{-\ \ 9+288-2268} \\ -32+252 \end{array}$$

Try 9 as another,

Then, $x^2-32x+252=0$ is the equation which will give the other roots. Solving, x is found equal to 14 and 18. $\therefore$ 7, 9, 14, 18 are the roots.

REVIEW EXERCISES.

Page 367.

1. The equation may be expressed,

$$\frac{9x}{2}-\frac{7}{2}+\frac{11}{x}+2x+6=\frac{14}{x}+\frac{75x}{6}-\frac{7}{2}-\frac{3}{x}. \qquad (1)$$

Simplifying, $$\frac{48x}{6}=2x+6. \qquad (2)$$

Clearing, $$48x=12x+36; \qquad (3)$$

whence, $$x=1.$$

2. Clearing, $$bx-b^2x+x^3=x^3-2bx-x^2+2b+2bx-2b. \qquad (1)$$

Transposing, etc., $$x^2=b^2x-bx. \qquad (2)$$

Dividing by x, $$x=b^2-b, \text{ or } b(b-1).$$

3. The equation may be expressed,

$$ax+\frac{bx}{4}+a-ab^2=bx+a-\frac{3bx}{a}+\frac{3bx}{a}-\frac{5a}{2}. \qquad (1)$$

Simplifying, $$ax+\frac{bx}{4}-ab^2=bx-\frac{5a}{2}. \qquad (2)$$

Clearing, $$4ax+bx-4ab^2=4bx-10a. \qquad (3)$$

Collecting terms, $$(4a-3b)x=2a(2b^2-5); \qquad (4)$$

whence, $$x=\frac{2a(2b^2-5)}{4a-3b}.$$

4. Clearing, $6x + 6 + 4x + 8 = 192 - 3x - 9.$ (1)

Transposing, etc., $13x = 169;$ (2)

whence, $x = 13.$

5. The equation may be expressed,

$$1 - \frac{14}{x+7} + \frac{1}{2(x+7)} = 1 - \frac{9}{2x-6}. \qquad (1)$$

Simplifying, $\frac{27}{2(x+7)} = \frac{9}{2x-6}.$ (2)

Clearing, $54x - 162 = 18x + 126;$ (3)

whence, $x = 8.$

6. The equation may be expressed,

$$\frac{5x}{9} - \frac{4}{9} + \frac{12x+2}{11x-8} = \frac{5x}{9} + \frac{17}{18}. \qquad (1)$$

Simplifying, $\frac{12x+2}{11x-8} = \frac{25}{18}.$ (2)

Clearing, $216x + 36 = 275x - 200;$ (3)

whence, $x = 4.$

7. Clearing, $x + 3 + \sqrt{x^2 + 3x} = 6.$ (1)

Transposing, etc., $\sqrt{x^2 + 3x} = 3 - x.$ (2)

Squaring, $x^2 + 3x = 9 - 6x + x^2.$ (3)

Transposing, etc., $9x = 9;$ (4)

whence, $x = 1.$

8. Transposing, $\sqrt{x+5} = 7 - \sqrt{x+12}.$ (1)

Squaring, $x + 5 = 49 - 14\sqrt{x+12} + x + 12.$ (2)

Transposing, etc., $14\sqrt{x+12} = 56.$ (3)

Dividing by 14, $\sqrt{x+12} = 4.$ (4)

Squaring, etc., $x = 4.$

9. Clearing, $\sqrt{ax + x^2} + a + x = 2a.$ (1)

Transposing, etc., $\sqrt{ax + x^2} = a - x.$ (2)

Squaring, $ax + x^2 = a^2 - 2ax + x^2.$ (3)

Transposing, etc., $3ax = a^2;$ (4)

whence, $x = \frac{a}{3}.$

10. Clearing.

$2x^3+x^2-5x+2+3x^3-x^2-3x+1=5x^3+4x^2-23x-22.$ (1)

Transposing, etc., $4x^2-15x=25.$ (2)

Completing square, etc., $8x-15=\pm 25;$ (3)

whence, $x=5, \text{ or } -\frac{5}{4}.$

11. Rationalizing the denominator,

$$\frac{(\sqrt{a+x}+\sqrt{a-x})^2}{2x}=c. \qquad (1)$$

Clearing, $a+x+2\sqrt{a^2-x^2}+a-x=2cx.$ (2)

Transposing, etc., $2\sqrt{a^2-x^2}=2cx-2a.$ (3)

Dividing by 2, $\sqrt{a^2-x^2}=cx-a.$ (4)

Squaring, $a^2-x^2=c^2x^2-2acx+a^2.$ (5)

Transposing and dividing by x,

$$x+c^2x=2ac; \qquad (6)$$

whence, $$x=\frac{2ac}{1+c^2}.$$

Or, Art. 325, $$\frac{\sqrt{a+x}}{\sqrt{a-x}}=\frac{c+1}{c-1}.$$

Squaring, $$\frac{a+x}{a-x}=\frac{c^2+2c+1}{c^2-2c+1}.$$

Art. 325, $$\frac{x}{a}=\frac{2c}{c^2+1};$$

whence, $$x=\frac{2ac}{c^2+1}.$$

12. Clearing, $\sqrt{abx}+b=\sqrt{a^2x}-\sqrt{abx}-\sqrt{ab}+b.$ (1)

Transposing, etc., $2\sqrt{abx}-\sqrt{a^2x}=-\sqrt{ab}.$ (2)

Dividing by $-\sqrt{a}$, $\sqrt{ax}-2\sqrt{bx}=\sqrt{b}.$ (3)

Factoring, $(\sqrt{a}-2\sqrt{b})\sqrt{x}=\sqrt{b};$ (4)

whence, $$x=\frac{b}{(\sqrt{a}-2\sqrt{b})^2}.$$

13. Clearing, $\sqrt{b}-\sqrt{b-\sqrt{b^2-bx}}=c\sqrt{b}+c\sqrt{b-\sqrt{b^2-bx}}.$ (1)

Transposing, etc.,

$$-(c+1)\sqrt{b-\sqrt{b^2-bx}}=(c-1)\sqrt{b}. \qquad (2)$$

Squaring, $(c+1)^2b-(c+1)^2\sqrt{b^2-bx}=(c-1)^2b.$ (3)

Transposing etc., $(c+1)^2\sqrt{b^2-bx}=4bc.$ (4)

Squaring again, $(c+1)^4b^2-(c+1)^4bx=16b^2c^2.$ (5)

Transposing, etc., and dividing by $-b$,

$$(c+1)^4x = (c+1)^4b - 16\,bc^2; \qquad (6)$$

whence, $$x = \frac{b[(c+1)^4 - 16\,c^2]}{(c+1)^4};$$

or, $$= b\left[1 - \left(\frac{2\sqrt{c}}{c+1}\right)^4\right].$$

14. Squaring, $$a^2 + 2\,ax + x^2 = a^2 + x\sqrt{b^2 + x^2}. \qquad (1)$$

Transposing and dividing by x,

$$2\,a + x = \sqrt{b^2 + x^2}. \qquad (2)$$

Squaring again, $$4\,a^2 + 4\,ax + x^2 = b^2 + x^2. \qquad (3)$$

Transposing, etc., $$4\,ax = b^2 - 4\,a^2; \qquad (4)$$

whence, $$x = \frac{b^2 - 4\,a^2}{4\,a}.$$

Page 368.

15. In the first fraction divide numerator by denominator,

$$\sqrt{3\,x} - 1 = 1 + \frac{\sqrt{3\,x} - 1}{2}. \qquad (1)$$

Simplifying, $$\sqrt{3\,x} - 1 = 2. \qquad (2)$$

Transposing, $$\sqrt{3\,x} = 3; \qquad (3)$$

whence, $$x = 3.$$

16. Add 5 to each member,

$$x^2 - 2\,x + 5 + 6\sqrt{x^2 - 2\,x + 5} = 16. \qquad (1)$$

Completing square, etc.,

$$\sqrt{x^2 - 2\,x + 5} = 2, \text{ or } -8. \qquad (2)$$

Squaring, $$x^2 - 2\,x + 5 = 4, \text{ or } 64. \qquad (3)$$

Transposing, $$x^2 - 2\,x = -1, \text{ or } 59. \qquad (4)$$

Completing square, etc., $$x = 1, \text{ or } 1 \pm 2\sqrt{15}.$$

17. Clearing, $$x^3 - 40 = 3\sqrt{x^3}. \qquad (1)$$

Transposing, $$x^3 - 3\sqrt{x^3} = 40. \qquad (2)$$

Completing square, etc., $$\sqrt{x^3} = 8, \text{ or } -5. \qquad (3)$$

Squaring, $$x^3 = 64, \text{ or } 25; \qquad (4)$$

whence, $$x = 4, \text{ or } \sqrt[3]{25}.$$

18. Expanding the terms, we have,

$$9x^2 + 3x + \tfrac{1}{4} + 25x^2 - 5x + \tfrac{1}{4} = 54x^2 - 18x + \tfrac{3}{2} + 4x. \qquad (1)$$

Transposing, etc., $20x^2 - 12x = -1.$ (2)

Completing square, etc., $x = \frac{1}{2}$, or $\frac{1}{10}$.

19. The expression becomes,

$$5\sqrt{-1} + 7\sqrt{-1} + 11\sqrt{-1} - 8\sqrt{-1} + \sqrt{-1} - 6\sqrt{-1} \pm 0\sqrt{-2} = 10\sqrt{-1}.$$

20. $(4) \times 2 - (3)$, $16y + 4x = 48.$ (5)

$(1) \times 4 - (5) \times 3$, $-56y + 12z = -40.$ (6)

$(2) \times 6 - (6)$, $86y = 172;$ (7)

whence, $y = 2.$

Substituting, the other values are found,

$$x = 4;\ z = 6;\ u = 5.$$

21. $(1) + (2) \times 2$, $4x + 9y - z - 3u = 32.$ (6)

$(2) \times 2 + (3)$, $-4x + 14y + 5z + 5u = 26.$ (7)

$(3) + (4)$, $-8y + 11z + 4r = 16.$ (8)

$(5) + (8)$, $3x - 10y + 11z + 5u = 23.$ (9)

$(4) - (6)$, $-17y + 3z = -22.$ (10)

$(6) + (7)$, $23y + 4z + 2u = 58.$ (11)

$(7) \times 3 + (9) \times 4$, $2y + 59z + 35u = 170.$ (12)

$(12) \times 2 - (11) \times 35$, $-801y - 22z = -1690.$ (13)

$(13) \times 3 + (10) \times 22$, $-2777y = -5554;$ (14)

whence, $y = 2.$

Substituting, the other values are found,

$$x = 3;\ z = 4;\ u = -2;\ r = -3.$$

22. By Art. 325, $\dfrac{\sqrt{a^2 + x^2}}{x} = \dfrac{b + c}{b - c}.$ (2)

Squaring, $\dfrac{a^2 + x^2}{x^2} = \dfrac{b^2 + 2bc + c^2}{b^2 - 2bc + c^2}.$ (3)

By Art. 323, $\dfrac{a^2}{x^2} = \dfrac{4bc}{b^2 - 2bc + c^2}.$ (4)

Extracting root, $\dfrac{a}{x} = \dfrac{2\sqrt{bc}}{b - c};$ (5)

whence, $x = \pm \dfrac{a(b - c)}{2\sqrt{bc}}.$

23. Clearing, $3x - 4\sqrt{x} + 120\sqrt{x} - 160 = 15\sqrt{x} + 3x + 30 + 6\sqrt{x}.$ (1)

Transposing, etc., $95\sqrt{x} = 190;$ (2)

whence, $\sqrt{x} = 2;$ (3)

$$x = 4.$$

24. Clearing, $x\sqrt{a^2+x^2}+a^2+x^2=na^2.$ (1)

Transposing, $x\sqrt{a^2+x^2}=(n-1)a^2-x^2.$ (2)

Squaring, $a^2x^2+x^4=(n-1)^2a^4-2(n-1)a^2x^2+x^4.$ (3)

Transposing, etc., $(2a^2n-a^2)x^2=(n-1)^2a^4;$ (4)

whence, $x^2=\dfrac{(n-1)^2a^4}{a^2(2n-1)};$ (5)

$$x=\pm\frac{(n-1)a}{\sqrt{2n-1}}.$$

25. Clearing, $x^2-16-12x^2+300=12x^2-192.$ (1)

Transposing, etc., $23x^2=476;$ (2)

whence, $x=\pm\sqrt{\tfrac{476}{23}}=\pm 4.54+.$

26. Clearing, $d\sqrt{a^2-x^2}-d\sqrt{b^2+x^2}=c\sqrt{a^2-x^2}+c\sqrt{b^2+x^2}.$ (1)

Transposing, etc., $(d-c)\sqrt{a^2-x^2}=(d+c)\sqrt{b^2+x^2}.$ (2)

Squaring, $(d-c)^2a^2-(d-c)^2x^2=(d+c)^2b^2+(d+c)^2x^2.$ (3)

Transposing, etc., $2(d^2+c^2)x^2=(d-c)^2a^2-(d+c)^2b^2;$ (4)

whence, $x=\pm\sqrt{\dfrac{(d-c)^2a^2-(d+c)^2b^2}{2(d^2+c^2)}}.$

Page 369.

27. Clearing, $x\sqrt{a^2+x^2}+a^2+x^2=a^2+2.$ (1)

Transposing, etc., $x\sqrt{a^2+x^2}=2-x^2.$ (2)

Squaring, $a^2x^2+x^4=4-4x^2+x^4.$ (3)

Transposing, etc., $(a^2+4)x^2=4;$ (4)

whence, $x=\pm\dfrac{2}{\sqrt{a^2+4}}.$

28. Rationalizing the denominators of the first member, the equation becomes, $\dfrac{1+\sqrt{1-x^2}}{x^2}-\dfrac{1-\sqrt{1-x^2}}{x^2}=\dfrac{\sqrt{3}}{x^2}.$ (1)

Simplifying, $2\sqrt{1-x^2}=\sqrt{3}.$ (2)

Squaring, $4-4x^2=3.$ (3)

Transposing, etc., $4x^2=1;$ (4)

whence, $x=\pm\tfrac{1}{2}.$

29. $(2) \div 5,$ $\qquad x^2 + y^2 = 13.$ (3)

$(1) - (3) \times 4,$ $\qquad xy = 6.$ (4)

$(4) \times 2,$ $\qquad 2xy = 12.$ (5)

$(3) + (5),$ $\qquad x^2 + 2xy + y^2 = 25.$ (6)

$(3) - (5),$ $\qquad x^2 - 2xy + y^2 = 1.$ (7)

Taking square root (6), $\qquad x + y = \pm 5.$ (8)

Taking square root (7), $\qquad x - y = \pm 1.$ (9)

Uniting (8) and (9), $\qquad x = \pm 3, \pm 2;\ y = \pm 2, \pm 3.$

30. Clearing, the first equation becomes, $xy = ab.$ (1)

$(2) + \sqrt{(1)} \times 2,$ $\qquad x + 2\sqrt{xy} + y = a + 2\sqrt{ab} + b.$ (3)

$(2) - \sqrt{(1)} \times 2,$ $\qquad x - 2\sqrt{xy} + y = a - 2\sqrt{ab} + b.$ (4)

Taking square root (3), $\qquad \sqrt{x} + \sqrt{y} = \pm(\sqrt{a} + \sqrt{b}).$ (5)

Taking square root (4), $\qquad \sqrt{x} - \sqrt{y} = \pm(\sqrt{a} - \sqrt{b}).$ (6)

Uniting (5) and (6), $\qquad x = a,$ or $b;\ y = b,$ or $a.$

31. Dividing (1) by $x + y,$ $\qquad x^2 - xy + y^2 = xy.$ (1)

Transposing and taking square root, $x - y = 0.$ (3)

Uniting (2) and (3), $\qquad x = 2;\ y = 2.$

32. Let $x + y = m,$ and $x - y = n;$ the equations become,

$$mn - m = 8; \tag{1}$$

$$mn^2 = 32. \tag{2}$$

From (1), $\qquad m = \dfrac{8}{n-1}.$

From (2), $\qquad m = \dfrac{32}{n^2};$

whence, $\qquad \dfrac{8}{n-1} = \dfrac{32}{n^2}.$ (3)

Solving (3), $\qquad n = 2;\ m = 8.$

Therefore, $\qquad x + y = 8;$ and $x - y = 2.$

Solving, $\qquad x = 5;\ y = 3.$

33. Squaring (1), $\qquad \dfrac{1}{x^2} + \dfrac{2}{xy} + \dfrac{1}{y^2} = \tfrac{1}{4}.$ (3)

Subtracting $(2) \times 2,$ $\qquad \dfrac{1}{x^2} - \dfrac{2}{xy} + \dfrac{1}{y^2} = \tfrac{1}{36}.$ (4)

Extracting square root of (4), $\qquad \dfrac{1}{x} - \dfrac{1}{y} = \pm \tfrac{1}{6}.$ (5)

From (5) and (1), $\qquad x = 3,$ or $6;\ y = 6,$ or $3.$

34. From (1), $$\sqrt{x}=\frac{21-\sqrt{y}}{y}. \qquad (3)$$

From (2), $$x=\frac{333-y}{y^2}. \qquad (4)$$

Equating (3) and (4)², $$\frac{333-y}{y^2}=\frac{441-42\sqrt{y}+y}{y^2}. \qquad (5)$$

Simplifying, $$2y-42\sqrt{y}=-108. \qquad (6)$$

Dividing (6) by 2, and completing square, etc.,

$$y=9, \text{ or } 324.$$

Substituting, $$x=4, \text{ or } (\tfrac{1}{108})^2.$$

35. From (1), $$y^{\frac{2}{3}}=3x-x^{\frac{3}{2}}. \qquad (3)$$

From (2), $$y^{\frac{1}{3}}=x-x^{\frac{1}{2}}. \qquad (4)$$

Equating (3) and (4)², $$3x-x^{\frac{3}{2}}=x^2-2x^{\frac{3}{2}}+x. \qquad (5)$$

Transposing and dividing by x, $$x-x^{\frac{1}{2}}=2. \qquad (6)$$

Completing square, etc., $$\sqrt{x}=2, \text{ or } -1;$$

$$x=4, \text{ or } 1.$$

Substituting, $$y=8.$$

36. (2)³, $$x^3-3x^2y+3xy^2-y^3=\frac{4096}{x^3y^3}. \qquad (3)$$

(3) − (1), $$-3x^2y+3xy^2=\frac{4096}{x^3y^3}-56. \qquad (4)$$

Clearing (2), and multiplying by 3,

$$3x^2y-3xy^2=48. \qquad (5)$$

(4) + (5), $$\frac{4096}{x^3y^3}=8. \qquad (6)$$

From (6), $$x^3y^3=512; \qquad (7)$$

whence, $$xy=8.$$

Substituting (7) in (2), $$x-y=2. \qquad (8)$$

Uniting (7) and (8), $x=4$, or -2; $y=2$, or -4.

37. (1) × 6 y − (2), $$6y^3-y^4=72y-432. \qquad (3)$$

(3) may be written, $$6y^3+432=y^4+72y. \qquad (4)$$

Factoring, $$6(y^3+72)=y(y^3+72); \qquad (5)$$

whence, $$y=6, \text{ or } -2\sqrt[3]{9}.$$

Substituting in (1), $$x=2, \text{ or } \sqrt[3]{3}-\sqrt[3]{9}.$$

38. (1) + (2), $x^2 + 2xy + y^2 + 4x + 4y = 117.$ (3)
(3) may be written, $(x + y)^2 + 4(x + y) = 117.$ (4)
Completing square, etc., $x + y = 9, \text{ or } -13.$ (5)
Eliminating x from (2) and (5), $y^2 + 2y = 35, \text{ or } 57.$ (6)
Solving, $y = 5, \; -7, \; -1 \pm \sqrt{58};$
$x = 4, \; 16, \; -12 \mp \sqrt{58}.$

39. Transposing (1) and extracting the square root,
$x + x^2y = \pm 21.$ (3)
(3) − (2) × x, $x^2 + 4x = \pm 21.$ (4)
Solving, $x = 3, \; -7, \; -2 \pm \sqrt{-17};$
$y = 2, \; \frac{4}{7}, \; \frac{1}{7}(5 \mp \sqrt{-17}).$

40. The first equation may be expressed,
$4(x - 2y)^2 + x - 2y = 5.$ (3)
Completing the square, etc., $x - 2y = 1, \text{ or } -\frac{5}{4}.$ (4)
From (4), $x = 1 + 2y, \text{ or } -\frac{5}{4} + 2y.$ (5)
Substituting (5) in (2), and solving, $y = 1, \; -\frac{7}{8}, \; \frac{5}{6} \pm \frac{1}{12}\sqrt{409};$
$x = 3, \; -\frac{11}{8}, \; \frac{5}{12} \pm \frac{1}{6}\sqrt{409}.$

41. From (1), $y = \frac{19 - 2x}{3}.$
Substituting in (2),
$45x^2 - 2527 + 532x - 28x^2 = 558.$ (3)
Collecting terms, etc., $17x^2 + 352x = 3085.$ (4)
Solving, $x = 5, \text{ or } -36\frac{5}{17};$
$y = 3, \text{ or } 30\frac{9}{17}.$

42. From (1), by completing square, $xy = 8.$ (3)
Squaring (2), $x^2 + 2xy + y^2 = 36.$ (4)
(4) − (3) × 4, $x^2 - 2xy + y^2 = 4.$ (5)
Extracting square root, $x - y = \pm 2;$ (6)
$x = 4, \; 2, \; 3 \pm \sqrt{21};$
$y = 2, \; 4, \; 3 \mp \sqrt{21}.$

43. (1) + (2) × 3, $x^3 + 3x^2y + 3xy^2 + y^3 = 125.$ (3)
Taking cube root, $x + y = 5.$ (4)
Factoring (2), $xy(x + y) = 30.$ (5)
Substituting (4) in (5), $xy = 6;$ (6)
whence, $x = 3, \text{ or } 2;$
$y = 2, \text{ or } 3.$

44. (1) − (2), $xy - y = 2.$ (3)

From (3), $y = \dfrac{2}{x-1}.$ (4)

Substituting (4) in (2), and collecting terms,

$10x^2 - 26x = -12.$ (5)

Dividing by 2, $5x^2 - 13x = -6.$ (6)

Completing square, etc., $x = 2, \text{ or } \frac{3}{5}.$

Substituting in (4), $y = 2, \text{ or } -5.$

45. Factoring (2), $xy(x^2 + y^2) = 160.$ (3)

Substituting (1) in (3), $xy = 8.$ (4)

(1) + (4) × 2, $x^2 + 2xy + y^2 = 36.$ (5)

(1) − (4) × 2, $x^2 - 2xy + y^2 = 4.$ (6)

Taking square root (5), $x + y = \pm 6.$ (7)

Taking square root (6), $x - y = \pm 2.$ (8)

Uniting (7) and (8), $x = \pm 4, \text{ or } \pm 2;\ y = \pm 2, \text{ or } \pm 4.$

46. Let $x = vy;$

then (1) becomes, $v^4y^4 - y^4 = 65.$ (3)

(2) becomes, $v^3y^4 + vy^4 = 78.$ (4)

From (3), $y^4 = \dfrac{65}{v^4 - 1}.$ (5)

From (4), $y^4 = \dfrac{78}{v^3 + v}.$ (6)

Equating (5) and (6), $\dfrac{65}{v^4 - 1} = \dfrac{78}{v(v^2 + 1)}.$ (7)

Solving (7), $v = \frac{3}{2}, \text{ or } -\frac{2}{3}.$

Substituting, $y = \pm 2,\ \pm 3\sqrt[4]{-1};\ x = \pm 3,\ \mp 2\sqrt[4]{-1}.$

47. (1) + (2) × 2, $(x + y)^2 + x + y = 240.$ (3)

Completing square, etc., $x + y = 15, \text{ or } -16.$ (4)

Substituting in (2), $xy = 56, \text{ or } 87.$ (5)

From (4) and (5), $x = 8,\ 7,\ -8 \pm \sqrt{-23};\ y = 7,\ 8,\ -8 \mp \sqrt{-23}.$

48. The first equation may be expressed,

$(x + y)^2 + 2(x + y) = 120.$ (1)

Completing square, etc., $x + y = 10, \text{ or } -12.$ (3)

From (2), $x = \dfrac{8 + y^2}{y}.$ (4)

From (3), $x = 10 - y, \text{ or } -12 - y.$ (5)

Equating (4) and (5), and solving, $y = 4,\ 1,\ -3 \pm \sqrt{5}.$

Substituting, $x = 6,\ 9,\ -9 \mp \sqrt{5}.$

Page 370.

49. From (1), $32y^2 = 18x^2.$ (3)

Dividing by 2, and taking square root, $4y = \pm 3x.$ (4)

From (2), $y = \frac{48}{x}.$ (5)

From (4), $y = \pm \frac{3}{4}x.$ (6)

Equating (6) and (5), and solving, $x = \pm 8, \pm 8\sqrt{-1}.$

Substituting, $y = \pm 6, \mp 6\sqrt{-1}.$

50. Completing square, etc., in (1), $x + y = 9$, or $16.$ (3)

Cubing (3), $x^3 + 3x^2y + 3xy^2 + y^3 = 729$, or $4096.$ (4)

(4) − (2), $3x^2y + 3xy^2 = 540$, or $3907.$ (5)

Dividing by 3, and factoring, $xy(x + y) = 180$, or $\frac{3907}{3}.$ (6)

Substituting (3) in (6), $xy = 20$, or $\frac{3907}{48}.$ (7)

From (3) and (7), $x = 5, 4, 8 \pm \frac{1}{12}\sqrt{-2505}$;

$y = 4, 5, 8 \mp \frac{1}{12}\sqrt{-2505}.$

51. (1) − (2), $10x + 2y = 14.$ (3)

From (3), $y = 7 - 5x.$ (4)

Expanding (1),

$21x^2 + 22xy - 8y^2 + 3x + 4y = 44.$ (1)

Substituting (4) in (1),

$21x^2 + 154x - 110x^2 - 392 + 560x - 200x^2 + 3x + 28 - 20x = 44.$ (5)

Collecting terms, etc., $17x^2 - 41x = -24.$ (6)

Completing square, etc., $x = 1$, or $1\frac{7}{17}.$

Substituting in (4), $y = 2$, or $-\frac{1}{17}.$

52. From (1), $xy = 36.$ (3)

Squaring (2), $x + 2\sqrt{xy} + y = 25.$ (4)

From (3), $\sqrt{xy} = \pm 6.$

Substituting in (4), $x + y = 13$, or $37.$ (5)

From (3) and (5), $x = 9, 4, 36, 1$; $y = 4, 9, 1, 36.$

53. Clearing (1), $x^3 + y^3 = 9xy.$ (3)

Raising (1) to 3d power, $x^3 + 3x^2y + 3xy^2 + y^3 = 216.$ (4)

(4) − (3), $3x^2y + 3xy^2 = 216 - 9xy.$ (5)

Dividing by 3, and factoring, $xy(x + y) = 72 - 3xy.$ (6)

Substituting (2) in (6), $xy = 8.$ (7)

From (2) and (7), $x = 4$, or 2; $y = 2$, or $4.$

54. Squaring (1), $(x^4 + 2x^2y^2 + y^4)x^2y^2 = 90000.$ (3)

Substituting (2) in (3), $2x^4y^4 + 337x^2y^2 = 90000.$ (4)

Completing square, etc., $xy = \pm 12.$ (5)

Substituting (5) in (1), $x^2 + y^2 = \pm 25.$ (6)

(6) + (5) × 2 (taking root), $x + y = \pm 7$, or ± 1. (7)

(6) − (5) × 2 (taking root), $x - y = \pm 1$, or ± 7. (8)

Uniting (7) and (8),

$$x = 4,\ -4,\ 3,\ -3,\ 3\sqrt{-1},\ -3\sqrt{-1},\ 4\sqrt{-1},\ -4\sqrt{-1};$$
$$y = \pm 3,\ \pm 3,\ \pm 4,\ \pm 4,\ \pm 4\sqrt{-1},\ \pm 4\sqrt{-1},\ \pm 3\sqrt{-1},\ \pm 3\sqrt{-1}.$$

By (4), xy has also two imaginary values, $\pm 12\sqrt{-2}$. But the student will not be able to find the corresponding roots.

55. Multiplying both terms of the fraction in (1) by $x - y$,

$$\frac{x^3 - y^3}{x^2 - y^2} = 14. \qquad (3)$$

Multiplying both terms of the fraction in (2) by $x + y$,

$$\frac{x^3 + y^3}{x^2 - y^2} = 18. \qquad (4)$$

(3) + (4), $$\frac{2x^3}{x^2 - y^2} = 32. \qquad (5)$$

(4) − (3), $$\frac{2y^3}{x^2 - y^2} = 4. \qquad (6)$$

(5) ÷ (6), $$\frac{x^3}{y^3} = 8;\quad \therefore x = 2y; \qquad (7)$$

whence, $y = 6$ and $x = 12$.

56. Equations (1) and (2) may be expressed as follows:

$$\frac{x^4 + x^2y^2 + y^4}{y^2} = 84. \qquad (1)$$

$$\frac{x^2 + xy + y^2}{y} = 14. \qquad (2)$$

(1) ÷ (2), $$\frac{x^2 - xy + y^2}{y} = 6. \qquad (3)$$

(2) − (3), $$\frac{2xy}{y},\ \text{or } 2x = 8; \qquad (4)$$

whence, $x = 4.$

Substituting in (3), $y = 8$, or 2.

57. (1) + (2) × 2, $(x + y)^2 + 3(x + y) = 40.$ (3)

Completing square, etc., $x + y = 5$, or -8. (4)

Substituting in (2), $xy = 6$, or 32. (5)

From (4) and (5), $x = 3,\ 2,\ -4 \pm 4\sqrt{-1}$; $y = 2,\ 3,\ -4 \mp 4\sqrt{-1}$.

58. (2) ÷ (1), $2(x^2 - y^2)(x - y) = 3xy.$ (3)

(1) − (3), $4xy^2 + 4x^2y = 12xy.$ (4)

(4) ÷ 4 xy, $x + y = 3.$ (5)

(1) + (3), $4x^3 + 4y^3 = 18xy.$ (6)

Dividing by 2, $2x^3 + 2y^3 = 9xy.$ (7)

$(5)^3 \times 2 - (7)$, $6x^2y + 6xy^2 = 54 - 9xy.$ (8)

Dividing by 3, and factoring, $2xy(x + y) = 18 - 3xy.$ (9)

Substituting (5) in (9), $xy = 2.$ (10)

From (5) and (10), $x = 2$, or 1; $y = 1$, or 2.

59. (2) − (1) × 4, $11x^2 = 176;$ (3)

whence, $x = \pm 4.$

Substituting in (1), $(y \pm 4)^2 = 81;$ (4)

whence, $y = \pm 5$, or ∓ 13.

60. Squaring (1), $(x^2 + 2xy + y^2)x^2y^2 = 900.$ (3)

(3) − (2), $(2xy)x^2y^2 = 432.$ (4)

From (4), $x^3y^3 = 216.$ (5)

Taking cube root, $xy = 6.$ (6)

Substituting (6) in (1), $x + y = 5.$ (7)

From (6) and (7), $x = 3$, or 2; $y = 2$, or 3.

61. Putting $u + v$ for x, and $u - v$ for y,

(1) becomes, $10u^2 + 10v^2 + 4u^2 - 4v^2 = 356.$ (3)

(2) becomes, $2u^2 + 2v^2 + 2u = 62.$ (4)

From (3), $14u^2 + 6v^2 = 356.$ (5)

(5) − (4) × 3, $8u^2 - 6u = 170.$ (6)

Solving (6), $u = 5$, or $-\frac{17}{4}$.

Substituting in (5), $v = \pm 1$, or $\pm \frac{5}{4}\sqrt{11}$.

$\therefore x = 6, 4, \frac{1}{4}(-17 \pm 5\sqrt{11});\ y = 4, 6, \frac{1}{4}(-17 \mp 5\sqrt{11}).$

62. Expanding (2), $x^5 + x^3y^2 + x^2y^3 + y^5 = 455.$ (2)

Raising (1) to 5th power,

$x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5 = 3125.$ (3)

(3) − (2), $5x^4y + 9x^3y^2 + 9x^2y^3 + 5xy^4 = 2670.$ (4)

$(1)^3 \times 5xy$, $5x^4y + 15x^3y^2 + 15x^2y^3 + 5xy^4 = 625xy.$ (5)

(5) − (4), $6x^3y^2 + 6x^2y^3 = 625xy - 2670.$ (6)

Factoring (6), $6x^2y^2(x + y) = 625xy - 2670.$ (7)

Substituting (1) in (7), $30x^2y^2 = 625xy - 2670.$ (8)

Dividing by 5, $6x^2y^2 - 125xy = -534.$ (9)

Completing square, etc., $xy = 6$, or $\frac{89}{6}$. (10)

From (1) and (10), $x = 3, 2, \frac{5}{2} \pm \frac{1}{6}\sqrt{-309};\ y = 2, 3, \frac{5}{2} \mp \frac{1}{6}\sqrt{-309}.$

63. (1) + (2), $(x+y)^2 - 24(x+y) = -140.$ (3)
Completing square, etc., $x + y = 14$, or 10. (4)
Substituting in (1), $xy = 48$, or 0. (5)
From (4) and (5), $x = 8, 6, 10, 0;\ y = 6, 8, 0, 10.$

64. (1) + (2) + (3), $(x+y+z)(x+y+z) = 225.$ (4)
Taking square root of (4), $x + y + z = \pm 15.$ (5)
Substituting (5) in (1), (2), and (3), successively,
$x = \pm 4;\ y = \pm 5;\ z = \pm 6.$

65. (1) − (2), $y^2 - y = 56.$ (4)
Completing square, etc., $y = 8$, or -7. (5)
Substituting (5) in (3), and reducing, $x + z = 11$, or $-\frac{103}{7}$. (6)
Substituting (5) in (2), $x^2 + z^2 = 61$, or 76. (7)
$(6)^2 - (7)$, $2xz = 60$, or $\frac{6885}{49}$. (8)
Uniting (6) and (8),
$x = 6, 5, \frac{1}{14}(-103 \pm \sqrt{-3161});\ y = 8, 8, -7, -7;$
$z = 5, 6, \frac{1}{14}(-103 \mp \sqrt{-3161}).$

Page 371.

66. Raising (1) to 3d power, and subtracting (2),
$3x^2y + 3xy^2 = a^3 - b.$ (3)
Factoring (3), $3xy(x+y) = a^3 - b.$ (4)
Substituting (1) in (4), and reducing, $xy = \frac{a^3 - b}{3a}.$ (5)
$(1)^2 - (5) \times 4$, $x^2 - 2xy + y^2 = \frac{4b - a^3}{3a}.$ (6)
Taking square root, $x - y = \pm\sqrt{\frac{4b - a^3}{3a}}.$ (7)
Uniting (1) and (7), $x = \frac{a}{2} \pm \frac{1}{2}\sqrt{\frac{4b - a^3}{3a}};$
$y = \frac{a}{2} \mp \frac{1}{2}\sqrt{\frac{4b - a^3}{3a}}.$

67. (1) − (2), $x - y = a - b..$ (3)
From (3), $y = x - a + b.$ (4)
Substituting (4) in (1), $x + \sqrt{x^2 - ax + bx} = a.$ (5)
Transposing (5), and squaring, $x^2 - ax + bx = a^2 - 2ax + x^2;$ (6)
whence, $x = \frac{a^2}{a+b}.$
Substituting in (4), $y = \frac{b^2}{a+b}.$

68. Expanding (1), $x^3 - x^2y - xy^2 + y^3 = a.$ (1)

Expanding (2), $x^3 + x^2y + xy^2 + y^3 = b.$ (2)

(1) + (2), (divided by 2), $x^3 + y^3 = \frac{a+b}{2}.$ (3)

(2) − (1), (divided by 2), $x^2y + xy^2 = \frac{b-a}{2}.$ (4)

(3) + (4) × 3, $x^3 + 3x^2y + 3xy^2 + y^3 = 2b - a.$ (5)

Taking cube root of (5), $x + y = \sqrt[3]{2b-a}.$ (6)

Factoring (4), $xy(x+y) = \frac{b-a}{2}.$ (7)

Substituting (6) in (7), $xy = \frac{b-a}{2\sqrt[3]{2b-a}}.$ (8)

$(6)^2 - (8) \times 4$, $x^2 - 2xy + y^2 = \frac{a}{\sqrt[3]{2b-a}}.$ (9)

Taking square root, $x - y = \frac{\sqrt{a}}{\sqrt[6]{2b-a}}.$ (10)

Uniting (6) and (10),

$$y = \frac{\sqrt{2b-a} \mp \sqrt{a}}{2\sqrt[6]{2b-a}};$$

$$x = \frac{\sqrt{2b-a} \pm \sqrt{a}}{2\sqrt[6]{2b-a}}.$$

69. $a = \frac{1}{2};\ l = \frac{2}{3};\ n = 8.$

Formula, $l = a + (n-1)d.$

Substituting, $\frac{2}{3} = \frac{1}{2} + 7d;$

whence, $d = \frac{1}{42};\ \frac{1}{2} + \frac{1}{42} = \frac{11}{21};\ \frac{11}{21} + \frac{1}{42} = \frac{23}{42};\ \frac{23}{42} + \frac{1}{42} = \frac{4}{7};$
$\frac{4}{7} + \frac{1}{42} = \frac{25}{42};\ \frac{25}{42} + \frac{1}{42} = \frac{13}{21};\ \frac{13}{21} + \frac{1}{42} = \frac{9}{14}.$

$\therefore \frac{11}{21}, \frac{23}{42}, \frac{4}{7}, \frac{25}{42}, \frac{13}{21}, \frac{9}{14}$ are the terms.

70. $a = 9;\ l = \frac{1}{9};\ n = 5.$

Formula, $l = ar^{n-1}.$

Substituting, $\frac{1}{9} = 9r^4;$

whence, $r = \frac{1}{3};\ 9 \times \frac{1}{3} = 3;\ 3 \times \frac{1}{3} = 1;\ 1 \times \frac{1}{3} = \frac{1}{3}.$

$\therefore 3, 1, \frac{1}{3}$ are the terms.

71. $a = \frac{8}{5};\ l = -\frac{10}{7};\ n = 3.$

Formula, $l = a + (n-1)d.$

Substituting, $-\frac{10}{7} = \frac{8}{5} + 2d;$

whence, $d = -\frac{53}{35};\ \frac{8}{5} + \left(-\frac{53}{35}\right) = \frac{3}{35}.$

72. The geometrical mean between two terms is the square root of their product, $\sqrt{23\frac{1}{3} \times 4\frac{2}{7}} = \sqrt{100} = 10.$

73. $(a-x)^{\frac{1}{2}} = a^{\frac{1}{2}} - \frac{1}{2}a^{-\frac{1}{2}}x + \left(\frac{-\frac{1}{2}\cdot\frac{1}{2}}{1\cdot 2}\right)a^{-\frac{3}{2}}x^2 - \left(\frac{-\frac{1}{2}\cdot\frac{1}{2}\cdot-\frac{3}{2}}{1\cdot 2\cdot 3}\right)a^{-\frac{5}{2}}x^3$

$+\left(\frac{-\frac{1}{2}\cdot\frac{1}{2}\cdot-\frac{3}{2}\cdot-\frac{5}{2}}{1\cdot 2\cdot 3\cdot 4}\right)a^{-\frac{7}{2}}x^4 = a^{\frac{1}{2}} - \frac{1}{2}a^{-\frac{1}{2}}x - \frac{1}{8}a^{-\frac{3}{2}}x^2 - \frac{1}{16}a^{-\frac{5}{2}}x^3 - \frac{5}{128}a^{-\frac{7}{2}}x^4.$

74. $(a^3+b^2)^{n+1} = (a^3)^{n+1} + (n+1)(a^3)^n b^2 + \left(\frac{n(n+1)}{1\cdot 2}\right)(a^3)^{n-1}b^4$

$+\left(\frac{n(n+1)(n-1)}{1\cdot 2\cdot 3}\right)(a^3)^{n-2}b^6 + \left(\frac{n(n+1)(n-1)(n-2)}{1\cdot 2\cdot 3\cdot 4}\right)(a^3)^{n-3}b^8$

$= a^{3n+3} + (n+1)a^{3n}b^2 + \frac{n(n+1)}{2}a^{3n-3}b^4 + \frac{n(n+1)(n-1)}{6}a^{3n-6}b^6$

$+\frac{n(n+1)(n-1)(n-2)}{24}a^{3n-9}b^8.$

75. $(1+2x-x^2)^3 = [(1+2x)-x^2]^3 = (1+2x)^3 - 3(1+2x)^2x^2$
$+3(1+2x)x^4 - x^6 = 1 + 6x + 12x^2 + 8x^3 - 3x^2 - 12x^3 - 12x^4$
$+3x^4 + 6x^5 - x^6 = 1 + 6x + 9x^2 - 4x^3 - 9x^4 + 6x^5 - x^6.$

76. The fifth term $= +\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}(a^{-1})^6(2x^{\frac{1}{2}})^4$
$= +3360\,a^{-6}x^2.$

77. $\sqrt[4]{125} = (11^2+2^2)^{\frac{1}{4}} = (11^2)^{\frac{1}{4}} + \frac{1}{4}(11^2)^{-\frac{3}{4}}\cdot 2^2 - \frac{3}{32}(11^2)^{-\frac{7}{4}}\cdot 2^4$
$= 11^{\frac{1}{2}} + \frac{1}{11^{\frac{3}{2}}} - \frac{3\cdot 16}{32\cdot 11^{\frac{7}{2}}} = 3.3166 + .0274 - .0003 = 3.3437+.$

78. Assume,
$$\frac{1-x}{1-3x-2x^2} = A + Bx + Cx^2 + Dx^3 + Ex^4 \ldots$$

Clearing,
$$\begin{array}{rcrrrrr} 1-x & = & A+ & Bx+ & Cx^2+ & Dx^3+ & Ex^4 \\ & & & -3A & -3B & -3C & -3D. \\ & & & & -2A & -2B & -2C. \end{array}$$

Equating coefficients of like powers of x,
$$A=1;\ B-3A=-1;\ C-3B-2A=0;$$
$$D-3C-2B=0;\ E-3D-2C=0;$$
whence, $\quad A=1;\ B=2;\ C=8;\ D=28;\ E=100.$
$$\therefore \frac{1-x}{1-3x-2x^2} = 1 + 2x + 8x^2 + 28x^3 + 100x^4.$$

79. Assume,
$$\frac{7x^4+1}{x^8-1} = \frac{A}{x^4+1} + \frac{B}{x^4-1}.$$

Clearing,
$$7x^4+1 = Ax^4 - A + Bx^4 + B;$$
$$= (A+B)x^4 - A + B.$$

Equating coefficients of equal powers of x,
$$A+B=7;\ -A+B=1;$$
whence, $\quad A=3;\ B=4.$
$$\therefore \frac{7x^4+1}{x^8-1} = \frac{3}{x^4+1} + \frac{4}{x^4-1}.$$

80. Assume, $\dfrac{5x^2+10x+4}{x^3+3x^2+2x}=\dfrac{A}{x}+\dfrac{B}{x+2}+\dfrac{C}{x+1}.$

Clearing, $5x^2+10x+4=Ax^2+3Ax+2A+Bx^2+Bx+Cx^2+2Cx;$

$=(A+B+C)x^2+(3A+B+2C)x+2A.$

Equating coefficients of like powers of x,

$A+B+C=5;\ 3A+B+2C=10;\ 2a=4;$

whence, $A=2;\ B=2;\ C=1.$

$$\therefore \frac{5x^2+10x+4}{x^3+3x^2+2x}=\frac{2}{x}+\frac{2}{x+2}+\frac{1}{x+1}.$$

83. If 5 is a root, $x-5$ is a factor, and the equation may be written,

$(x-5)(x^2-6x+7)=0;$

whence, $x^2-6x+7=0;$

and, $x^2-6x=-7.$

Solving, $x=3\pm\sqrt{2}$, the other roots.

84. The equation is

$(x-1)(x+2)(x+4)=0.$

Expanding, $x^3+5x^2+2x-8=0.$

Page 372.

85. (2) × 7 − (1) × 5, $41x+3y=200.$ (3)

From (3), $y=\dfrac{200-41x}{3}.$ (4)

Changing (4) to mixed quantity, $y=66-14x+\dfrac{2+x}{3}.$ (5)

Since the values are to be in whole numbers, the second member of (5) must be a whole number.

Therefore, $\dfrac{2+x}{3}=w$(whole number);

whence, $x=3w-2.$

Giving w different values beginning with 1,

$x=1, 4, 7, 10, 13, 16,$ etc.

Substituting, $y=53, 12, -29, -70,$ etc.;

$z=-24, 4, 32, 60,$ etc.

The only positive values which satisfy the original equations are 4, 12, 4.

86. Let $x=$ one part;

$590-x=$ the other part;

then, $(590-x)x=80464.$ (1)

Expanding, $x^2-590x=-80464.$ (2)

Completing square, etc., $x=376$, first part;

$590-x=214$, second part.

87. Reducing $\frac{2}{3}$, 6, $\frac{7}{8}$ to a common denominator, we have $\frac{16}{24}$, $\frac{144}{24}$, $\frac{21}{24}$. Therefore, the three parts are proportional to 16, 144, and 21.

Let $16x =$ the first part;

then, $144x =$ the second part;

and, $\underline{21x} =$ the third part;

$$181x = 327;$$

$$x = 1\tfrac{146}{181};$$

$16x = 28\frac{164}{181}$, the first part;

$144x = 260\frac{28}{181}$, the second part;

$21x = 37\frac{170}{181}$, the third part.

88. Let $x =$ the cost;

and, $\dfrac{x}{100} =$ the gain per cent;

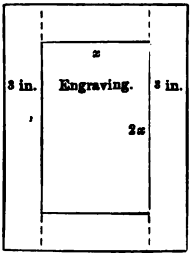
then, $$x + \frac{x^2}{100} = 144. \qquad (1)$$

Clearing, $$x^2 + 100x = 14400. \qquad (2)$$

Completing square, etc., $x = 80$ dollars, the cost.

89. Let $x =$ breadth of engraving;

$2x =$ length of engraving.



The area of the margin is $18x + 36$ inches, and the area of the engraving is $2x^2$ inches.

Therefore, from the conditions,

$$2x^2 - 36 = 18x + 36.$$

Solving, $x = 12$ inches, the breadth.

90. Let $x =$ the length in feet;

$y =$ the breadth in feet;

$z =$ the height in feet;

then, $$x : y :: 5 : 4;\ \text{or,}\ 4x = 5y; \qquad (1)$$

$$z : y :: 7 : 8;\ \text{or,}\ 8z = 7y; \qquad (2)$$

and, $$xy^2z = 224\,xy. \qquad (3)$$

From (3), $$yz = 224. \qquad (4)$$

$(4) \times 8 - (2) \times y$, $$7y^2 = 1792. \qquad (5)$$

Solving, $y = 16$ feet, breadth.

Substituting, $x = 20$ feet, length;

$z = 14$ feet, height.

91. Let $x =$ the depth of the pit;

$\frac{x}{1100} =$ the time for sound to come back;

$7 - \frac{x}{1100}$, or $\frac{7700 - x}{1100} =$ the time for stone to fall.

The square of the time multiplied by 16 equals the distance.

Therefore, $$16\left(\frac{7700 - x}{1100}\right)^2 = x. \quad (1)$$

Expanding and clearing,

$$59290000 - 15400x + x^2 = 75625x. \quad (2)$$

Transposing, etc., $$x^2 - 91025x = -59290000. \quad (3)$$

Solving (3), $x = 90368.9 +$ feet, or $656.1 -$ feet.

As the first value is too large to fulfill the conditions, the second value, 656.1 feet, is the depth of the pit.

92. Let $x =$ length of hypotenuse;

$x - 24 =$ length of base;

$x - 3 =$ length of perpendicular.

The square on the hypotenuse equals the sum of the squares on the other two sides.

Therefore,

$$x^2 - 48x + 576 + x^2 - 6x + 9 = x^2. \quad (1)$$

Uniting, $$x^2 - 54x = -585.$$

Solving, $x = 39$ feet, hypotenuse;

$x - 24 = 15$ feet, base;

$x - 3 = 36$ feet, perpendicular.

93. Let $10x + y =$ the less number he sold;

$10y + x =$ the greater number he sold;

then, $$147 - (10y + x) = \frac{147 - (10x + y)}{2}. \quad (1)$$

Simplifying, $$8x - 19y = -147. \quad (2)$$

From (2), $$x = \frac{19y - 147}{8}; \quad (3)$$

or, $$x = 2y - 18 + \frac{3y - 3}{8}. \quad (4)$$

Place $\frac{3y - 3}{8} = w$ (a whole number);

whence, $$y = \frac{8w + 3}{3}.$$

Assigning integral values to w, and solving,

$$y = 9;\ x = 3.$$

∴ The numbers are 39, from first flock of sheep; 93, from second flock of sheep.

Page 373.

94. Let $x =$ the capacity of the third vessel.

If x gallons of wine are taken out of the cask holding a gallons, and x gallons of water are put in, the cask will contain a gallons of the mixture of which $a - x$ gallons are wine; therefore, the proportion of wine in the mixture is $\frac{a-x}{a}$.

If x gallons of water are taken from the cask containing b gallons, and x gallons of wine are put in, the cask will then contain b gallons of which x gallons are wine; therefore, the proportion of wine in the mixture is $\frac{x}{b}$.

By the conditions the proportions in the mixtures are equal.

Therefore, $$\frac{a-x}{a} = \frac{x}{b};$$

whence, $$x = \frac{ab}{a+b}.$$

95. Let $x =$ the hypotenuse;

$y =$ the perpendicular;

then, $$\sqrt{xy} = 20; \quad (1)$$

and, $$x^2 - y^2 = 400. \quad (2)$$

From (1), $$xy = 400; \quad (3)$$

whence, $$y = \frac{400}{x}. \quad (4)$$

Substituting in (2), $$x^2 - \frac{160000}{x^2} = 400. \quad (5)$$

Solving, $x = 25.4$ inches, hypotenuse.

96. Let x, y, and $z =$ the numbers;

then, $$(y - x) - (z - y) = 6; \quad (1)$$

$$x + y + z = 33; \quad (2)$$

$$x^2 + y^2 + z^2 = 441. \quad (3)$$

Simplifying (1), $$-x + 2y - z = 6. \quad (4)$$

(2) + (4), $$3y = 39;$$

whence, $$y = 13. \quad (5)$$

Substituting (5) in (2), $$x + z = 20. \quad (6)$$

Substituting (5) in (3), $x^2 + z^2 = 272.$ (7)

$(6)^2 - (7),$ $2\,xz = 128.$ (8)

Solving (6) and (8), $x = 16$, or 4; $z = 4$, or 16.

∴ 4, 13, 16 are the numbers.

97. Let $x = \text{length};\ y = \text{breadth};$

$$\frac{7\,x}{8} = \text{length after wetting};$$

$$\frac{15\,y}{16} = \text{breadth after wetting};$$

$$2\,x + 2\,y = \text{perimeter before wetting};$$

$$xy = \text{area before wetting};$$

$$2\left(\frac{14\,x + 15\,y}{16}\right) = \text{perimeter after wetting};$$

$$\frac{105\,xy}{108} = \text{area after wetting};$$

then, $$2\,x + 2\,y - 4\tfrac{1}{2} = 2\left(\frac{14\,x + 15\,y}{16}\right); \quad (1)$$

$$xy - 5\tfrac{3}{4} = \frac{105\,xy}{128}. \quad (2)$$

Simplifying (1), $2\,x + y = 34.$ (3)

Simplifying (2), $xy = 32.$ (4)

From (4), $y = \frac{32}{x}.$ (5)

Substituting in (3), $2\,x + \frac{32}{x} = 34.$ (6)

Solving, $x = 16$ feet, length;

$y = 2$ feet, breadth.

98. x = number of feet added to breadth;

y = number of feet taken from length;

119 feet × 2 + 19 feet × 2 = 276 feet = perimeter of the original lot;

119 × 19 = 2261 square feet, the area of the original lot;

then, $2(119 - y) + 2(x + 19) = 300;$ (1)

$(119 - y)(x + 19) = 2261.$ (2)

(1) becomes, $x - y = 12.$ (3)

(2) becomes, $119\,x - 19\,y - xy = 0.$ (4)

From (3), $y = x - 12.$ (5)

From (4), $y = \frac{119x}{19 + x}$. (6)

Equating (5) and (6), and solving,

$x = 114$, feet added to breadth;

$y = 102$, feet taken from length.

99. Let $x =$ miles A traveled per day;

$y =$ miles B traveled per day;

$x - y =$ number days they traveled;

then, $x^2 - xy = 216$; (1)

$xy - y^2 = 180$. (2)

(1) – (2), $x^2 - 2xy + y^2 = 36$. (3)

Extracting square root of (3), $x - y = 6$. (4)

Factoring (1), $x(x - y) = 216$. (5)

Substituting (4) in (5), $x = 36$, miles A went per day;

$y = 30$, miles B went per day.

100. Let x and $y =$ the numbers;

then, $x + y = xy$; (1)

$x + y = x^2 - y^2$. (2)

Dividing (2) by $x + y$, $1 = x - y$. (3)

From (3), $x = 1 + y$. (4)

From (1), $x = \frac{-y}{1 - y}$. (5)

Equating (4) and (5), and clearing,

$1 - y^2 = -y$. (6)

Solving, $y = \frac{1}{2}(1 \pm \sqrt{5})$;

$x = \frac{1}{2}(3 \pm \sqrt{5})$.

101. Let $x =$ one number; $y =$ the other;

then, $xy = x^2 - y^2$; (1)

$x^2 + y^2 = x^3 - y^3$. (2)

(2) – (1) × 2, $x^2 - 2xy + y^2 = x^3 - y^3 - 2(x^2 - y^2)$. (3)

Dividing by $x - y$, $x - y = x^2 + xy + y^2 - 2(x + y)$. (4)

From (4), $3x + y = x^2 + xy + y^2$. (5)

(5) + (1), $3x + y = 2x^2$. (6)

From (6), $y = 2x^2 - 3x$. (7)

Substituting (7) in (1), $2x^3 - 3x^2 = x^2 - 4x^4 + 12x^3 - 9x^2$. (8)

Collecting terms and dividing by $2x^2$,

$$2x^2 - 5x = -\tfrac{5}{2}. \qquad (9)$$

Solving (9), $x = \tfrac{1}{4}(5 \pm \sqrt{5})$.

Substituting in (7), $y = \tfrac{1}{2}\sqrt{5}$.

Page 374.

102. Let $x =$ the side of the greater;

$y =$ the side of the less;

then, $x^3y + xy^3 = 820;$ (1)

$x^2 - y^2 = 9.$ (2)

From (1), $x^2 + y^2 = \dfrac{820}{xy}.$ (3)

Squaring (3), $x^4 + 2x^2y^2 + y^4 = \dfrac{672400}{x^2y^2}.$ (4)

Squaring (2), $x^4 - 2x^2y^2 + y^4 = 81.$ (5)

(4) − (5), $4x^2y^2 = \dfrac{672400}{x^2y^2} - 81.$ (6)

Clearing, etc., $4x^4y^4 + 81x^2y^2 = 672400.$ (7)

Completing square, etc., $xy = 20.$ (8)

Substituting value of xy in (3),

$x^2 + y^2 = 41.$ (9)

(9) + (2), $x^2 = 25;$

$x = 5;$

and, $y = 4.$

Therefore, $x^3y = 500$ dollars, the price of greater;

and, $xy^3 = 320$ dollars, the price of less.

103. Let $x =$ length of upper part;

$y =$ length of remaining part;

then, $4x + 6y = 5x + 5y + 28;$ (1)

$x : y :: 9 : 16.$ (2)

From (1), $-x + y = 28.$ (3)

From (2), $16x = 9y.$ (4)

Solving (3) and (4), $x = 36;\ y = 64.$

∴ 36 + 64, or 100 feet, is the entire height.

104. Let $x =$ distance the second messenger had traveled when they met;
$y =$ distance the first messenger had traveled when they met;
and, $x + y =$ distance between the two towns;
$\frac{x}{2\frac{1}{12}}$, or $\frac{12x}{25} =$ rate per hour of second messenger;
$\frac{y}{4\frac{1}{12}}$ or $\frac{12y}{49} =$ rate per hour of first messenger.

Then, since each completed the journey in the same time,

$$\frac{y}{\frac{12x}{25}} = \frac{x}{\frac{12y}{49}}. \qquad (1)$$

Simplifying,
$$\frac{25y}{12x} = \frac{49x}{12y}. \qquad (2)$$

Clearing,
$$25y^2 = 49x^2; \qquad (3)$$

whence,
$$5y = 7x; \qquad (4)$$

$$y = \frac{7x}{5}.$$

Therefore, the entire distance is $x + \frac{7x}{5}$, or $\frac{12x}{5}$.

The entire distance divided by the rate of each will give the time in which each will perform the journey.

$$\frac{12x}{5} \div \frac{12x}{25} = 5 \text{ hours, time for second messenger.}$$

$$\frac{12x}{5} \div \left(\frac{12}{49} \times \frac{7x}{5}\right) = 7 \text{ hours, time for first messenger.}$$

105. Let $x =$ the number of days the voyage was expected to last;
then, $x + 21 =$ the number of days the voyage lasted;
and, $x+21-30$ or $x-9=$the number of days after the scurvy broke out.

Since 3 men died per day, on the 31st day 172 men were alive, on the 32d, 169 men, etc., consequently the number of men alive on the last day is the last term of an arithmetical progression of which the first term is 172, the number of terms $x - 9$ and the common difference 3, or

$$\text{Last term} = 172 - 3(x - 10).$$

Since each man received a ration of water per day, the whole number of rations of water issued after the scurvy appeared was equal to the sum of an arithmetical series whose first term is 172, last term, $172-3(x-10)$, number of terms, $x - 9$, or,

$$\text{The sum} = (344 - 3x + 30)\frac{x-9}{2}.$$

And that number of rations of water was equal to the number allowed to 175 men for $x - 30$ days, or $175(x - 30)$.

$$\therefore\ (344 - 3x + 30)\frac{x-9}{2} = 175(x - 30).$$

Simplifying, $x^2 - 17x = 2378;$ (2)

whence, $x = 58;$

and, $x + 21 = 79$ days, length of voyage;

also, $172 - 3(x - 10) = 28$, the number of men alive.

106. Let x = length; y = breadth;

then, $(x + 7)(y + 4) = xy + 363;$ (1)

$(x - 4)(y - 3) = xy - 208.$ (2)

From (1), $4x + 7y = 335.$ (3)

From (2), $-3x - 4y = -220.$ (4)

$(3) \times 3 + (4) \times 4$, $5y = 125;$ (5)

whence, $y = 25$ feet, breadth;

$x = 40$ feet, length.

107. Let x = number gallons in greater;

y = number gallons in less;

$y - 6$ = smaller quantity after drawing 6 gallons;

$x - 6$ = larger quantity after drawing 6 gallons;

then, $y - 6 : x - 6 :: 4 : 7$, or $7y - 4x = 18;$ (1)

$y - 3 : x - 2 :: 7 : 12$, or $12y - 7x = 22.$ (2)

$(1) \times 7 - (2) \times 4$, $y = 38$, gallons in less;

$x = 62$, gallons in greater.

Page 375.

108. Let x represent the number of miles A or B travels per hour.

$\frac{3}{2}$ miles per hour is the rate of the geese.

$\frac{9}{4}$ miles per hour is the rate of the wagon.

A meets the wagon $50 - 2x$ miles from Baltimore, and B meets the wagon $31 + \frac{2x}{3}$ miles from Baltimore. Therefore, the wagon went $\frac{8x}{3} - 19$ miles in the meantime.

Since the wagon goes $\frac{9}{4}$ miles per hour, the time required to move $\frac{8x}{3} - 19$ miles is equal to $\left(\frac{8x}{3} - 19\right) \div \frac{9}{4}$, or $\frac{4}{9}\left(\frac{8x}{3} - 19\right)$ hours.

Since A travels x miles per hour *toward* Baltimore, and the wagon moves $\frac{9}{4}$ miles per hour *from* Baltimore, the distance between A and the wagon is increased $x + \frac{9}{4}$ miles per hour; and since B met the wagon $\frac{4}{9}\left(\frac{8x}{3} - 19\right)$ hours after A met it, $\frac{4}{9}\left(\frac{8x}{3} - 19\right)\left(x + \frac{9}{4}\right)$ is equal to the distance between A and B at that time.

Since the geese had traveled 5 miles farther at the rate of $\frac{3}{2}$ miles per hour before B overtook them, it took them $\frac{10}{3}$ hours to travel the distance. And since A was traveling during that time at the rate of x miles per hour, he must have been $\frac{10x}{3}$ miles ahead of the geese, lacking the 5 miles which the geese had traveled in that time, or $\frac{10x}{3} - 5$ miles; and A is $\frac{10x}{3} - 5$ miles ahead of B when B overtakes the geese.

Since A and B travel at the same rate, as soon as both begin to travel, they continue the same distance apart.

Therefore, $\frac{4}{9}\left(\frac{8x}{3} - 19\right)\left(x + \frac{9}{4}\right) = \frac{10x}{3} - 5.$

Solving, $x = 9.$

And, since they were always the same distance apart, when A reached Baltimore B was $\frac{10x}{3} - 5$ miles, or 25 miles distant.

109. If the area of the pond is $\frac{1}{9}$ of the whole garden, its sides are $\frac{1}{3}$ the sides of the garden.

Let $3x =$ length of garden;

then, $x =$ length of pond.

Let $3y =$ breadth of garden;

then, $y =$ breadth of pond;

then, $6x + 6y - 2x - 2y = 200;$ (1)

$(3x + 3)(3y + 5) = 9xy + 645.$ (2)

From (1), $x + y = 50.$ (3)

From (2), $5x + 3y = 210.$ (4)

$(4) + (3) \times 3,$ $2x = 60;$ (5)

whence, $x = 30;$

$y = 20.$

Therefore, $3x = 90$ yards, length of garden;

$3y = 60$ yards, breadth of garden.

110. Let $x =$ number of men at first;

then, $x - 42 =$ number of men probably alive on eighth day, if there had been no sally;

and, $\frac{2x - 42}{2}$, or $x - 21 =$ average number alive daily for eight days.

Therefore, $8x - 168 =$ number of days' provisions on hand at first;

and, $\frac{2x - 30}{2} \times 6$, or $6x - 90 =$ days' provisions exhausted at end of sixth day.

Then, from conditions of the problem,

$$8x - 168 - (6x - 90) = 366;$$

whence, $x = 222$.

$222 - (100 + 36) = 86$, number of men left after the sally.

Let $y =$ number of days the provisions lasted afterwards.

$86 =$ number living after the sally.

$86 - 10(y - 1) =$ number living on last day provisions lasted.

Therefore, $\frac{172 - 10(y - 1)}{2} =$ average number living per day;

and, $\frac{172-10(y-1)}{2} \cdot y =$ whole number of days' provisions left after the sally.

Consequently, $\frac{172-10(y-1)}{2} \cdot y = 366$.

Solving, $y = 6$;

and, $86 - 60$, or $26 =$ number of men left when provisions were exhausted.

111. Let $x =$ the height of the wall;

$8x - 2 =$ the length of yard;

$6x - 5 =$ the breadth of yard;

then, $(8x - 2)(6x - 5) - 2x(8x - 2) - 2x(6x - 5) = 178$. (1)

Simplifying (1), $10x^2 - 19x = 84$. (2)

Solving (2), $x = 4$ yards, height of wall;

$8x - 2 = 30$ yards, length of yard;

$6x - 5 = 19$ yards, breadth of yard.

112. $\sqrt{\frac{3}{7}} = \sqrt{.42857142} + = .6546 + = .655$.

113. $\sqrt{355} = 18.841443 +$; (1)

$\sqrt{113} = 10.630145 +$. (2)

(1) + (2), $29.471588 +$. (3)

(1) − (2), $8.211298 +$. (4)

Square root of (3), $5.428 +$. (5)

Square root of (4), $2.865 +$. (6)

(5) ÷ (6), 1.895.